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著者	Ito Tatsuro, Terwillinger Paul
journal or publication title	Kyushu Journal of Mathematics
volume	64
number	1
page range	81-144
year	2010-06-22
URL	http://hdl.handle.net/2297/25168

doi: 10.2206/kyushujm.64.81

The augmented tridiagonal algebra

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Dedicated to Professor Eiichi Bannai on the occasion of his retirement

Abstract

Motivated by investigations of the tridiagonal pairs of linear transformations, we introduce the augmented tridiagonal algebra \mathcal{T}_q . This is an infinite-dimensional associative \mathbb{C} -algebra with 1. We classify the finite-dimensional irreducible representations of \mathcal{T}_q . All such representations are explicitly constructed via embeddings of \mathcal{T}_q into the $U_q(sl_2)$ -loop algebra. As an application, tridiagonal pairs over \mathbb{C} are classified in the case where q is not a root of unity.

Keywords. P- and Q-polynomial association scheme, Terwilliger algebra, tridiagonal pair, Leonard pair, tridiagonal algebra, q -Onsager algebra.

2000 Mathematics Subject Classification. Primary: 17B37. Secondary: 05E30, 33D45.

1 Introduction

The purpose of this paper is to introduce the augmented tridiagonal algebra \mathcal{T}_q and classify its finite-dimensional irreducible representations. We explain our motivations in Sections 1.1, 1.2 and summarize our results in Sections 1.3, 1.4. Throughout this paper, we choose the complex number field \mathbb{C} as the ground field. An algebra means an associative \mathbb{C} -algebra with 1.

1.1 Tridiagonal pairs: a background in combinatorics

The standard generators for the subconstituent algebra (Terwilliger algebra) of a P- and Q-polynomial association scheme [1] give rise to a tridiagonal pair when they are restricted to an irreducible submodule of the standard module [3, Example 1.4], [9, Lemmas 3.9, 3.12]. This fact motivates our ongoing investigation of the tridiagonal pairs [3], [4], [5], [6], [7], [8].

Let V denote a finite-dimensional nonzero vector space over \mathbb{C} . Let $\text{End}(V)$ denote the \mathbb{C} -algebra of all \mathbb{C} -linear transformations of V . By a *tridiagonal pair* (*TD-pair*) on V we mean an ordered pair A, A^* of elements in $\text{End}(V)$ that satisfy (i)–(iv) below:

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(i) A and A^* are diagonalizable.

(ii) There exists an ordering V_0, V_1, \dots, V_d of the eigenspaces of A such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0, V_{d+1} = 0$.

(iii) There exists an ordering $V_0^*, V_1^*, \dots, V_{d^*}^*$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq d^*),$$

where $V_{-1}^* = 0, V_{d^*+1}^* = 0$.

(iv) V is irreducible as an $\langle A, A^* \rangle$ -module, where $\langle A, A^* \rangle$ is the subalgebra of $\text{End}(V)$ generated by A, A^* .

A TD-pair $A, A^* \in \text{End}(V)$ is *isomorphic* to a TD-pair $B, B^* \in \text{End}(V')$ whenever there exists an isomorphism $\psi : V \rightarrow V'$ of vector spaces such that $B\psi = \psi A$ and $B^*\psi = \psi A^*$.

In this subsection, we summarize the basic properties of a TD-pair A, A^* ; they will be used to introduce the augmented tridiagonal algebra \mathcal{T}_q in the next subsection. First we remark that A and A^* have the same number of eigenvalues, i.e. $d = d^*$ [2, Lemma 4.5]: we call this common integer the *diameter* of the pair. A TD-pair with $d = 0$ is called *trivial*. We usually assume $d \geq 1$ to avoid the trivial TD-pairs. Under this assumption, there exist exactly two orderings of the eigenspaces of A (resp. A^*) that satisfy the condition (ii) (resp. (iii)): if V_0, V_1, \dots, V_d (resp. $V_0^*, V_1^*, \dots, V_{d^*}^*$) is one of these, then the other is the reversed ordering V_d, V_{d-1}, \dots, V_0 (resp. $V_{d^*}^*, V_{d^*-1}^*, \dots, V_0^*$). We understand that one of such orderings is chosen and fixed unless otherwise stated.

By [3, Theorem 10.1] there exist scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ in \mathbb{C} such that

$$\begin{aligned} [A, A^2A^* - \beta AA^*A + A^*A^2] &= \gamma [A, AA^* + A^*A] + \delta [A, A^*], \\ [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2}] &= \gamma^* [A^*, A^*A + AA^*] + \delta^* [A^*, A], \end{aligned}$$

where $[X, Y]$ means $XY - YX$. The sequence of scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ is unique if $d \geq 3$. The above relations are called the *tridiagonal relations* (*TD-relations*) [8]. We fix a nonzero $q \in \mathbb{C}$ such that

$$\beta = q^2 + q^{-2}.$$

Let θ_i (resp. θ_i^*) denote the eigenvalue of A for V_i (resp. A^* for V_i^*) ($0 \leq i \leq d$). They are expressed as follows [3, Theorem 11.2].

Type I ($q^2 \neq \pm 1$): there exist scalars a, a^*, b, b^*, c, c^* such that

$$\begin{aligned} \theta_i &= a + bq^{2i} + cq^{-2i} & (0 \leq i \leq d), \\ \theta_i^* &= a + b^*q^{2i} + c^*q^{-2i} & (0 \leq i \leq d). \end{aligned}$$

In this case, $\gamma = -(q - q^{-1})^2 a, \gamma^* = -(q - q^{-1})^2 a^*, \delta = (q - q^{-1})^2 a^2 - (q^2 - q^{-2})^2 bc, \delta^* = (q - q^{-1})^2 a^{*2} - (q^2 - q^{-2})^2 b^*c^*$.

Type II ($q^2 = 1$): there exist scalars a, a^*, b, b^*, c, c^* such that

$$\begin{aligned}\theta_i &= a + b i + c i^2 & (0 \leq i \leq d), \\ \theta_i^* &= a^* + b^* i + c^* i^2 & (0 \leq i \leq d).\end{aligned}$$

In this case, $\gamma = 2c$, $\gamma^* = 2c^*$, $\delta = b^2 - c^2 - 4ac$, $\delta^* = b^{*2} - c^{*2} - 4a^*c^*$.

Type III ($q^2 = -1$): there exist scalars a, a^*, b, b^*, c, c^* such that

$$\begin{aligned}\theta_i &= a + b(-1)^i + c(-1)^i i & (0 \leq i \leq d), \\ \theta_i^* &= a^* + b^*(-1)^i + c^*(-1)^i i & (0 \leq i \leq d).\end{aligned}$$

In this case, $\gamma = 4a$, $\gamma^* = 4a^*$, $\delta = -4a^2 + c^2$, $\delta^* = -4a^{*2} + c^{*2}$.

In this paper, we treat TD-pairs of Type I. If a TD-pair of Type I comes from a P- and Q-polynomial association scheme with sufficiently large diameter, then q is not a root of unity, i.e., $q^n \neq 1$ for any nonzero integer n [1, Chapter 3, Proposition 7.7]. *From now on, we fix a nonzero scalar $q \in \mathbb{C}$ and assume that q is not a root of unity.* One of the effects of this assumption is as follows. Let us call the conditions (ii), (iii) for a TD-pair the *TD-structures*. Then under the diagonalizability condition (i) and the irreducibility condition (iv), the TD-relations imply the TD-structures [10, Theorem 3.10]. This allows us to work with the TD-relations instead of the TD-structures. We first establish the representation theory of the augmented tridiagonal algebra \mathcal{T}_q . The classification of TD-pairs of Type I will be given as an application of the representation theory.

If A, A^* are a TD-pair on V , then $\lambda A + \mu I$, $\lambda^* A^* + \mu^* I$ are also a TD-pair on V with the same eigenspaces. Here $\lambda, \lambda^*, \mu, \mu^* \in \mathbb{C}$, $\lambda \neq 0$, $\lambda^* \neq 0$ and I is the identity map. The parameter β and hence q are invariant under the affine transformations $A \mapsto \lambda A + \mu I$, $A^* \mapsto \lambda^* A^* + \mu^* I$. Also the diameter d is invariant under the affine transformations. For fixed d and q , consider a TD-pair A, A^* of Type I with diameter d and the parameter $\beta = q^2 + q^{-2}$. The TD-pair A, A^* can be standardized to have the following eigenvalues by applying appropriate affine transformations and, if necessary, reversing the ordering of the eigenspaces V_i of A or of the eigenspaces V_i^* of A^* :

$$\theta_i = b q^{2i-d} + \varepsilon b^{-1} q^{d-2i} \quad (0 \leq i \leq d), \quad (1)$$

$$\theta_i^* = \varepsilon^* b^* q^{2i-d} + b^{*-1} q^{d-2i} \quad (0 \leq i \leq d) \quad (2)$$

for some constants b, b^* ($b \neq 0$, $b^* \neq 0$) and $\varepsilon, \varepsilon^* \in \{1, 0\}$. A TD-pair A, A^* is called a *standardized* TD-pair of Type I, if A, A^* have eigenvalues $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$ as in (1), (2) respectively for some integer $d \geq 1$ and nonzero $b, b^* \in \mathbb{C}$ under suitable orderings of the eigenspaces $\{V_i\}_{i=0}^d$, $\{V_i^*\}_{i=0}^d$.

If $d = 1$, then θ_0, θ_1 (resp. θ_0^*, θ_1^*) can be any pair of distinct scalars by applying a suitable affine transformation to A (resp. A^*), in particular for standardization, (ε, b) and (ε^*, b^*) can be chosen arbitrarily from $\{0, 1\} \times \mathbb{C}^\times \setminus \{(1, \pm 1)\}$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Assume $d \geq 2$. Then the pair $\varepsilon, \varepsilon^*$ is uniquely determined by A, A^* regardless of standardization, but

the scalars b, b^* are not. If $\varepsilon = 1$ (resp. $\varepsilon^* = 1$), then b (resp. b^*) is determined up to the \pm sign by A (resp. A^*) and by the ordering of the eigenspaces of A (resp. A^*). In this case, b (resp. b^*) is changed to b^{-1} (resp. b^{*-1}) when we reverse the ordering of the eigenspaces of A (resp. A^*). If $\varepsilon = 0$ (resp. $\varepsilon^* = 0$), then b (resp. b^*) can be an arbitrary nonzero scalar. In this case, the ordering of the eigenspaces of A (resp. A^*) is uniquely determined when standardized.

If $(\varepsilon, \varepsilon^*) = (0, 1)$, we further standardize the TD-pair A, A^* by interchanging A, A^* and then reversing the ordering of the eigenspaces V_i^* so that the standardized TD-pair has $(\varepsilon, \varepsilon^*) = (1, 0)$. Thus a standardized TD-pair of Type I has

$$(\varepsilon, \varepsilon^*) = (1, 1), (1, 0) \text{ or } (0, 0)$$

and is called of the *1st, 2nd, 3rd kind*, accordingly.

The TD-relations for a standardized TD-pair A, A^* of Type I are

$$(TD) \begin{cases} [A, A^2 A^* - \beta A A^* A + A^* A^2] &= \varepsilon \delta [A, A^*], \\ [A^*, A^{*2} A - \beta A^* A A^* + A A^{*2}] &= \varepsilon^* \delta [A^*, A], \end{cases}$$

where $\beta = q^2 + q^{-2}$ and $\delta = -(q^2 - q^{-2})^2$. Conversely, if a TD-pair A, A^* satisfies the above TD-relations (TD), then we have $a = a^* = 0$, $bc = \varepsilon$, $b^*c^* = \varepsilon^*$ in the general expression of the eigenvalues for Type I, and so with suitable orderings of the eigenspaces, A, A^* have eigenvalues in the form of (1), (2) for some integer $d \geq 1$ and some nonzero b, b^* , i.e., A, A^* are a standardized TD-pair of Type I. Thus given $(\varepsilon, \varepsilon^*) \in \{(1, 1), (1, 0), (0, 0)\}$ and a nonzero scalar q that is not a root of unity, a TD-pair A, A^* is a standardized TD-pair if and only if it satisfies the above TD-relations (TD). In view of this fact, we call (TD) the *standardized* TD-relations of Type I.

Given TD-pair $A, A^* \in \text{End}(V)$ with eigenspaces $\{V_i\}_{i=0}^d$, $\{V_i^*\}_{i=0}^d$, the underlying vector space V has the *split decomposition* [3, Theorem 4.6]:

$$V = \bigoplus_{i=0}^d U_i,$$

where

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d).$$

For a TD-pair $A, A^* \in \text{End}(V)$ of Type I with eigenvalues (1), (2), let $K \in \text{End}(V)$ denote the diagonalizable transformation for which U_i is the eigenspace belonging to the eigenvalue q^{2i-d} ($0 \leq i \leq d$). We define the *raising map* R and the *lowering map* L by

$$\begin{aligned} R &= A - bK - \varepsilon b^{-1}K^{-1}, \\ L &= A^* - \varepsilon^* b^*K - b^{*-1}K^{-1}. \end{aligned}$$

Then indeed R (resp. L) has the *raising* (resp. *lowering*) *property* [3, Theorem 4.6, Corollary 6.3]:

$$\begin{aligned} RU_i &\subseteq U_{i+1} & (0 \leq i \leq d), \\ LU_i &\subseteq U_{i-1} & (0 \leq i \leq d), \end{aligned}$$

where $U_{-1} = U_{d+1} = 0$. By the raising, lowering properties of R, L , we get

$$(\text{TD})'_0 \begin{cases} KRK^{-1} &= q^2 R, \\ K L K^{-1} &= q^{-2} L, \end{cases}$$

and conversely the relations $(TD)'_0$ imply the raising, lowering properties of R, L . Writing $(TD)'_0$ in terms of A, A^*, K , we get the *generalized q -Weyl relations*:

$$(\text{TD})_0 \begin{cases} (qAK - q^{-1}KA)/(q - q^{-1}) &= bK^2 + \varepsilon b^{-1}I, \\ (qKA^* - q^{-1}A^*K)/(q - q^{-1}) &= \varepsilon^* b^* K^2 + b^{*-1}I, \end{cases}$$

where I is the identity map. Writing the tridiagonal relations (TD) for A, A^* in terms of R, L, K , we get

$$(\text{TD})' \begin{cases} [R, R^2L - \beta RLR + LR^2] &= \delta'(\varepsilon^* s^2 R^2 K^2 - \varepsilon s^{-2} K^{-2} R^2), \\ [L, L^2R - \beta LRL + RL^2] &= \delta'(-\varepsilon^* s^2 K^2 L^2 + \varepsilon s^{-2} L^2 K^{-2}), \end{cases}$$

where $\beta = q^2 + q^{-2}$, $\delta' = -(q - q^{-1})(q^2 - q^{-2})(q^3 - q^{-3})q^4$, $s^2 = b b^*$.

1.2 The TD-algebra \mathcal{A} and the augmented TD-algebra \mathcal{T}

Fix a nonzero scalar $q \in \mathbb{C}$ which is not a root of unity. We also fix $(\varepsilon, \varepsilon^*) \in \{(1, 1), (1, 0), (0, 0)\}$. Let $\mathcal{A} = \mathcal{A}_q^{(\varepsilon, \varepsilon^*)}$ denote the associative \mathbb{C} -algebra with 1 defined by generators z, z^* subject to the relations

$$(\text{TD}) \begin{cases} [z, z^2 z^* - \beta z z^* z + z^* z^2] &= \varepsilon \delta [z, z^*], \\ [z^*, z^{*2} z - \beta z^* z z^* + z z^{*2}] &= \varepsilon^* \delta [z^*, z], \end{cases}$$

where $\beta = q^2 + q^{-2}$ and $\delta = -(q^2 - q^{-2})^2$. When we need to specify $(\varepsilon, \varepsilon^*)$, we write $(\text{TD})_{\text{I}}, (\text{TD})_{\text{II}}, (\text{TD})_{\text{III}}$ for the relations (TD) and $\mathcal{A}_{\text{I}}, \mathcal{A}_{\text{II}}, \mathcal{A}_{\text{III}}$ for the algebra \mathcal{A} according to $(\varepsilon, \varepsilon^*) = (1, 1), (1, 0), (0, 0)$. The algebra \mathcal{A} is called the *tridiagonal algebra (TD-algebra)* [10] of the 1st, 2nd, 3rd kind, accordingly. $(\text{TD})_{\text{III}}$ is the q -Serre relations and \mathcal{A}_{III} is isomorphic to the positive part of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. $(\text{TD})_{\text{I}}$ can be regarded as a q -analogue of the Dolan-Grady relations and we call \mathcal{A}_{I} the *q -Onsager algebra*.

Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$ denote the associative \mathbb{C} -algebra with 1 defined by generators x, y, k, k^{-1} subject to the relations

$$(\text{TD})'_0 \begin{cases} k k^{-1} = k^{-1} k = 1, \\ k x k^{-1} = q^2 x, \\ k y k^{-1} = q^{-2} y, \end{cases}$$

and

$$(\text{TD})' \begin{cases} [x, x^2 y - \beta x y x + y x^2] = \delta'(\varepsilon^* x^2 k^2 - \varepsilon k^{-2} x^2), \\ [y, y^2 x - \beta y x y + x y^2] = \delta'(-\varepsilon^* k^2 y^2 + \varepsilon y^2 k^{-2}), \end{cases}$$

where $\beta = q^2 + q^{-2}$, $\delta' = -(q - q^{-1})(q^2 - q^{-2})(q^3 - q^{-3})q^4$. When we need to specify $(\varepsilon, \varepsilon^*)$, we write $(\text{TD})'_{\text{I}}, (\text{TD})'_{\text{II}}, (\text{TD})'_{\text{III}}$ for the relations $(\text{TD})'$ and $\mathcal{T}_{\text{I}}, \mathcal{T}_{\text{II}}, \mathcal{T}_{\text{III}}$ for the algebra \mathcal{T} according to $(\varepsilon, \varepsilon^*) = (1, 1), (1, 0), (0, 0)$. The algebra \mathcal{T} is called the *augmented tridiagonal*

algebra (augmented TD-algebra) of the 1st, 2nd, 3rd, kind, accordingly. \mathcal{T}_{III} is isomorphic to the Borel subalgebra of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

The augmented TD-algebra \mathcal{T} has another presentation. Fix a nonzero scalar $t \in \mathbb{C}$. Define the elements $z_t, z_t^* \in \mathcal{T}$ to be

$$z_t = x + t k + \varepsilon t^{-1} k^{-1}, \quad (3)$$

$$z_t^* = y + \varepsilon^* t^{-1} k + t k^{-1}. \quad (4)$$

Then \mathcal{T} is generated by z_t, z_t^*, k, k^{-1} and the following relations hold :

$$(TD)_0 \begin{cases} k k^{-1} = k^{-1} k = 1 \\ (q z_t k - q^{-1} k z_t)/(q - q^{-1}) = t k^2 + \varepsilon t^{-1}, \\ (q k z_t^* - q^{-1} z_t^* k)/(q - q^{-1}) = \varepsilon^* t^{-1} k^2 + t, \end{cases}$$

and

$$(TD) \begin{cases} [z_t, z_t^2 z_t^* - \beta z_t z_t^* z_t + z_t^* z_t^2] = \varepsilon \delta [z_t, z_t^*], \\ [z_t^*, z_t^{*2} z_t - \beta z_t^* z_t z_t^* + z_t z_t^{*2}] = \varepsilon^* \delta [z_t^*, z_t], \end{cases}$$

where $\beta = q^2 + q^{-2}$, $\delta = -(q^2 - q^{-2})^2$. One routinely verifies that \mathcal{T} is isomorphic to the algebra generated by symbols z_t, z_t^*, k, k^{-1} with $(TD)_0, (TD)$ the defining relations.

Proposition 1.1 *There exists an algebra homomorphism ι_t from \mathcal{A} to \mathcal{T} that sends z, z^* to z_t, z_t^* , respectively :*

$$\iota_t : \mathcal{A} \longrightarrow \mathcal{T} \quad (z, z^* \mapsto z_t, z_t^*).$$

Moreover ι_t is injective.

It is obvious that the correspondence $z, z^* \mapsto z_t, z_t^*$ can be extended to an algebra homomorphism from \mathcal{A} to \mathcal{T} . The injectivity of ι_t will be proved in Section 2.

Lemma 1.2 *Let $\rho : \mathcal{T} \longrightarrow \text{End}(V)$ be a finite-dimensional irreducible representation of \mathcal{T} . Then $\rho(k)$ is diagonalizable with eigenvalues $\{s q^{2i-d} \mid 0 \leq i \leq d\}$ for some nonzero $s \in \mathbb{C}$ and an integer $d \geq 0$. Let $V = \bigoplus_{i=0}^d U_i$ denote the eigenspace decomposition of $\rho(k)$, where U_i is the eigenspace belonging to $s q^{2i-d}$. Then regarding V as an irreducible \mathcal{T} -module via ρ , we have*

$$x U_i \subseteq U_{i+1}, \quad y U_i \subseteq U_{i-1} \quad (0 \leq i \leq d),$$

where $U_{-1} = U_{d+1} = 0$. In particular $\rho(x), \rho(y)$ are nilpotent.

The scalar s (resp. the integer d) is called the *type* (resp. *diameter*) of the representation ρ and the \mathcal{T} -module V . We call the direct sum $V = \bigoplus_{i=0}^d U_i$ the *weight-space decomposition* and U_0 the *highest weight space*.

Proof. For $\theta \in \mathbb{C}$, set $U(\theta) = \{v \in V \mid kv = \theta v\}$. Note that θ is an eigenvalue of $\rho(k)$ if and only if $U(\theta) \neq 0$, and in this case $U(\theta)$ is the corresponding eigenspace. Using the relations $kx = q^2 xk$ and $ky = q^{-2} yk$, we find $x U(\theta) \subseteq U(q^2 \theta)$ and $y U(\theta) \subseteq U(q^{-2} \theta)$. Now assume that θ is an eigenvalue of $\rho(k)$. Observe that $\theta \neq 0$ since k^{-1} exists, and that $\sum_{i \in \mathbb{Z}} U(q^{2i} \theta)$ is invariant under each of $x, y, k^{\pm 1}$ and the sum is a finite sum by $\dim V < \infty$. These elements $x, y, k^{\pm 1}$ generate \mathcal{T} and the \mathcal{T} -module V is irreducible, so we have $V = \sum_{i \in \mathbb{Z}} U(q^{2i} \theta)$. This yields the lemma. \square

Proposition 1.3 *Let $\rho : \mathcal{T} \longrightarrow \text{End}(V)$ be a finite-dimensional irreducible representation of \mathcal{T} with type s , diameter d , and $V = \bigoplus_{i=0}^d U_i$ the weight-space decomposition. Let z_t, z_t^* be as in (3), (4).*

(i) $\rho(z_t)$ is diagonalizable if and only if the scalars

$$\theta_i = stq^{2i-d} + \varepsilon s^{-1}t^{-1}q^{d-2i} \quad (0 \leq i \leq d)$$

are mutually distinct. In this case, $\{\theta_i\}_{i=0}^d$ is the set of eigenvalues of $\rho(z_t)$ and it holds that

$$V_i + V_{i+1} + \cdots + V_d = U_i + U_{i+1} + \cdots + U_d \quad (0 \leq i \leq d),$$

where V_i is the eigenspace of $\rho(z_t)$ belonging to θ_i .

(ii) $\rho(z_t^*)$ is diagonalizable if and only if the scalars

$$\theta_i^* = \varepsilon^* st^{-1}q^{2i-d} + s^{-1}tq^{d-2i} \quad (0 \leq i \leq d)$$

are mutually distinct. In this case, $\{\theta_i^*\}_{i=0}^d$ is the set of eigenvalues of $\rho(z_t^*)$ and it holds that

$$V_0^* + V_1^* + \cdots + V_i^* = U_0 + U_1 + \cdots + U_i \quad (0 \leq i \leq d),$$

where V_i^* is the eigenspace of $\rho(z_t^*)$ belonging to θ_i^* .

Proposition 1.3 will be proved in Section 2.

Recall we are given in advance $(\varepsilon, \varepsilon^*) \in \{(1, 1), (1, 0), (0, 0)\}$ and a nonzero scalar q that is not a root of unity. Let $\rho : \mathcal{A} \longrightarrow \text{End}(V)$ be a finite-dimensional irreducible representation of the TD-algebra $\mathcal{A} = \mathcal{A}_q^{(\varepsilon, \varepsilon^*)}$. We assume that ρ satisfies the following property (C₁):

(C₁): $\rho(z), \rho(z^*)$ are both diagonalizable.

Set $A = \rho(z)$, $A^* = \rho(z^*)$. Then A, A^* satisfy the TD-relations. The TD-relations for A, A^* imply the TD-structures, i.e., the conditions (ii), (iii) for a TD-pair hold for A, A^* , while the conditions (i), (iv) hold for A, A^* by the property (C₁) and the irreducibility of ρ . So $A, A^* \in \text{End}(V)$ are a TD-pair on V . Moreover since the TD-relations (TD) for A, A^* is the standardized TD-relations of Type I, the TD-pair A, A^* is a standardized TD-pair of Type I on V .

Conversely, let us start with a standardized TD-pair A, A^* of Type I on V , where we understand q and $(\varepsilon, \varepsilon^*)$ are chosen in advance and fixed. Consider the TD-algebra $\mathcal{A} = \mathcal{A}_q^{(\varepsilon, \varepsilon^*)}$. Then by the TD-relations (TD) for A, A^* , we obtain a finite-dimensional representation ρ of \mathcal{A} that sends z, z^* to A, A^* , respectively:

$$\rho : \mathcal{A} \longrightarrow \text{End}(V) \quad (z, z^* \mapsto A, A^*).$$

This representation ρ is irreducible and satisfies the property (C₁) by the conditions (iv), (i) for the TD-pair A, A^* .

We restate what we saw in the previous two paragraphs as a proposition below. We are given in advance $(\varepsilon, \varepsilon^*) \in \{(1, 1), (1, 0), (0, 0)\}$ and a nonzero scalar q that is not a root

of unity. Let \mathcal{STD} denote the set of isomorphism classes of standardized TD-pairs A, A^* of Type I together with the trivial TD-pairs: A (resp. A^*) has eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) as in (1) (resp. (2)) for some integer $d \geq 0$ and nonzero b (resp. b^*) $\in \mathbb{C}$ with a suitable ordering of the eigenspaces $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$). Set $\mathcal{A} = \mathcal{A}_q^{(\varepsilon, \varepsilon^*)}$. Let $\mathcal{Irr}(\mathcal{A})$ denote the set of isomorphism classes of finite-dimensional irreducible representations of \mathcal{A} that satisfy the property (C_1) . Then we have

Proposition 1.4 *The mapping $\rho \mapsto A = \rho(z), A^* = \rho(z^*)$ gives a bijection from $\mathcal{Irr}(\mathcal{A})$ to \mathcal{STD} . The trivial representations, i.e., 1-dimensional representations, correspond to the trivial TD-pairs.*

Thus the classification of standardized TD-pairs of Type I is reduced to the following problem.

Problem 1 *Classify up to isomorphism the finite-dimensional irreducible representations of \mathcal{A} that satisfy the property (C_1) .*

Let us start with a finite-dimensional irreducible representation $\rho : \mathcal{T} \longrightarrow \text{End}(V)$ of the augmented TD-algebra \mathcal{T} with type s and diameter d . We assume that ρ satisfies the following properties $(C_1)_t, (C_2)_t$ for some nonzero $t \in \mathbb{C}$:

- $(C_1)_t$: $\rho(z_t), \rho(z_t^*)$ are both diagonalizable.
- $(C_2)_t$: The restriction $\rho|_{\langle z_t, z_t^* \rangle} : \langle z_t, z_t^* \rangle \longrightarrow \text{End}(V)$ is irreducible, where $\langle z_t, z_t^* \rangle$ is the subalgebra of \mathcal{T} generated by z_t, z_t^* .

Set $A = \rho(z_t), A^* = \rho(z_t^*)$. Then A, A^* satisfy the TD-relations. Since the TD-relations for A, A^* imply the TD-structures for A, A^* , we find A, A^* are a TD-pair on V . By Proposition 1.3, the TD-pair A, A^* has distinct eigenvalues $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d$ as in (1), (2) with $b = st, b^* = st^{-1}$. So $A, A^* \in \text{End}(V)$ are a standardized TD-pair of Type I. By Lemma 1.2 and Proposition 1.3, the eigenspace decomposition for $\rho(k)$ coincides with the split decomposition for the TD-pair A, A^* . So we have $\rho(x) = R, \rho(y) = L, \rho(k) = sK$, where R, L, K are the raising, lowering, diagonalizable maps, respectively, associated with the split decomposition.

Conversely, let us start with a standardized TD-pair $A, A^* \in \text{End}(V)$ of Type I with eigenvalues

$$\begin{aligned} \theta_i &= bq^{2i-d} + \varepsilon b^{-1}q^{d-2i} & (0 \leq i \leq d), \\ \theta_i^* &= \varepsilon^* b^* q^{2i-d} + b^{*-1}q^{d-2i} & (0 \leq i \leq d), \end{aligned}$$

respectively as in (1), (2), where we understand q and $(\varepsilon, \varepsilon^*)$ are chosen in advance and fixed. We have the raising map R , the lowering map L and the diagonalizable K associated with the split decomposition for the TD-pair A, A^* . Consider the augmented TD-algebra $\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$. Define the nonzero scalars $s, t \in \mathbb{C}$ by

$$b = st, \quad b^* = st^{-1}. \tag{5}$$

The scalars s, t are determined by b, b^* up to the \pm sign : $s^2 = bb^*, t^2 = bb^{*-1}$. We choose s, t as one of the solutions of (5) and fix them. By the relations $(\text{TD})'_0, (\text{TD})'$ for R, L, K , we obtain a finite-dimensional representation ρ of \mathcal{T} with type s and diameter d that sends x, y, k to R, L, sK , respectively:

$$\rho : \mathcal{T} \longrightarrow \text{End}(V) \quad (x, y, k \mapsto R, L, sK)$$

expressed in terms of the 1st presentation of \mathcal{T} with respect to the generators x, y, k, k^{-1} . By (3), (4), it holds that $\rho(z_t) = A, \rho(z_t^*) = A^*$. So we have

$$\rho : \mathcal{T} \longrightarrow \text{End}(V) \quad (z_t, z_t^*, k \mapsto A, A^*, sK)$$

expressed in terms of the 2nd presentation of \mathcal{T} with respect to the generators z_t, z_t^*, k, k^{-1} . By the conditions (iv), (i) for the TD-pair A, A^* , ρ is irreducible and satisfies the properties $(C_1)_t, (C_2)_t$.

We restate what we saw in the previous two paragraphs as a proposition below. We are given in advance $(\varepsilon, \varepsilon^*) \in \{(1, 1), (1, 0), (0, 0)\}$ and a nonzero scalar q that is not a root of unity. Suppose that we are further given a positive integer d and nonzero $b, b^* \in \mathbb{C}$ such that the scalars $\theta_i = bq^{2i-d} + \varepsilon b^{-1}q^{d-2i}$ ($0 \leq i \leq d$) in (1) are mutually distinct and the scalars $\theta_i^* = \varepsilon^* b^* q^{2i-d} + b^{*-1}q^{d-2i}$ ($0 \leq i \leq d$) in (2) are mutually distinct. By $\mathcal{STD}_d^{(b, b^*)}$ we denote the set of isomorphism classes of standardized TD-pairs A, A^* of Type I with eigenvalues $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d$ respectively. Note that if a standardized TD-pair A, A^* of Type I belongs to $\mathcal{STD}_d^{(b, b^*)}$, then the ordering of the eigenspaces $\{V_i\}_{i=0}^d$ of A (resp. $\{V_i^*\}_{i=0}^d$ of A^*) is uniquely determined by the corresponding eigenvalues $\theta_i = bq^{2i-d} + \varepsilon b^{-1}q^{d-2i}$ (resp. $\theta_i^* = \varepsilon^* b^* q^{2i-d} + b^{*-1}q^{d-2i}$). Recall that if $\varepsilon = 1$ (resp. $\varepsilon^* = 1$), then b (resp. b^*) is changed to b^{-1} (resp. b^{*-1}) when we reverse the ordering of the eigenspaces of A (resp. A^*). Thus if $\varepsilon = 1$ (resp. $\varepsilon^* = 1$), then $\mathcal{STD}_d^{(b, b^*)} = \mathcal{STD}_d^{(b^{-1}, b^*)}$ (resp. $\mathcal{STD}_d^{(b, b^*)} = \mathcal{STD}_d^{(b, b^{*-1})}$): $\mathcal{STD}_d^{(b^{-1}, b^*)}$ (resp. $\mathcal{STD}_d^{(b, b^{*-1})}$) coincides with $\mathcal{STD}_d^{(b, b^*)}$ as sets of isomorphism classes of standardized TD-pairs A, A^* of Type I but has the ordering of the eigenspaces of A (resp. A^*) reversed. Set $b = st, b^* = st^{-1}$ as in (5). Such scalars s, t are determined by b, b^* uniquely up to the \pm sign. We choose one of them and fix it. Note that if (s, t) is a solution of $b = st, b^* = st^{-1}$, then

$$(s', t') = (t^{-1}, s^{-1}), (t, s), (s^{-1}, t^{-1}) \quad (6)$$

are a solution of $c = s't', c^* = s't'^{-1}$ for

$$(c, c^*) = (b^{-1}, b^*), (b, b^{*-1}), (b^{-1}, b^{*-1}) \quad (7)$$

respectively. Set $\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$. By $\mathcal{Irr}_d^{s, t}(\mathcal{T})$ we denote the set of isomorphism classes of finite-dimensional irreducible representations ρ of \mathcal{T} with type s and diameter d that satisfy the properties $(C_1)_t, (C_2)_t$ for the scalar t . Then we have

Proposition 1.5 *The mapping $\rho \mapsto A = \rho(z_t), A^* = \rho(z_t^*)$ gives a bijection from $\mathcal{Irr}_d^{s, t}(\mathcal{T})$ to $\mathcal{STD}_d^{(b, b^*)}$, where $b = st, b^* = st^{-1}$.*

Thus Problem 1 is reduced to the following problem.

Problem 2

- (i) *Classify up to isomorphism the finite-dimensional irreducible representations of \mathcal{T} with type s and diameter d .*
- (ii) *Determine when a finite-dimensional irreducible representation ρ of \mathcal{T} with type s and diameter d satisfies the properties $(C_1)_t$, $(C_2)_t$.*

We solve Problem 2 in this paper. Problem 1 is reduced to Problem 2 via $\mathcal{STD}_d^{(b,b^*)}$ as follows. The set \mathcal{STD} is the disjoint union of the trivial TD-pairs and $\mathcal{STD}_d^{(b,b^*)}$ over $d \in \mathbb{N}$ and $(b, b^*) \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) / \sim$, where \sim is the equivalence relation defined by $(b, b^*) \sim (c, c^*)$ if and only if

$$(c, c^*) \in \{(b, b^*), (b^{-1}, b^*), (b, b^{*-1}), (b^{-1}, b^{*-1})\} \quad \text{for the case } (\varepsilon, \varepsilon^*) = (1, 1), \quad (8)$$

$$(c, c^*) \in \{(b, b^*), (b^{-1}, b^*)\} \quad \text{for the case } (\varepsilon, \varepsilon^*) = (1, 0), \quad (9)$$

and $(b, b^*) = (c, c^*)$ for the case $(\varepsilon, \varepsilon^*) = (0, 0)$. For nonzero $b, b^* \in \mathbb{C}$, let $\mathcal{Irr}_d^{(b,b^*)}(\mathcal{A})$ denote the subset of $\mathcal{Irr}(\mathcal{A})$ that is mapped to the subset $\mathcal{STD}_d^{(b,b^*)}$ of \mathcal{STD} by the bijection $\rho \mapsto A = \rho(z)$, $A^* = \rho(z^*)$ from $\mathcal{Irr}(\mathcal{A})$ to \mathcal{STD} (see Proposition 1.4). Set $b = st$, $b^* = st^{-1}$ as in (5). Such scalars s, t are determined by b, b^* uniquely up to the \pm sign. We choose one of them and fix it. Then by Proposition 1.5, $\mathcal{Irr}_d^{s,t}(\mathcal{T})$ is mapped to $\mathcal{STD}_d^{(b,b^*)}$ by the bijection $\rho \mapsto A = \rho(z_t)$, $A^* = \rho(z_t^*)$. This means that if a finite-dimensional irreducible representation $\rho : \mathcal{T} \rightarrow \text{End}(V)$ belongs to $\mathcal{Irr}_d^{s,t}(\mathcal{T})$, then

$$\rho' = \rho \circ \iota_t : \mathcal{A} \rightarrow \text{End}(V)$$

is a finite-dimensional irreducible representation of \mathcal{A} that belongs to $\mathcal{Irr}_d^{(b,b^*)}(\mathcal{A})$, where

$$\iota_t : \mathcal{A} \rightarrow \mathcal{T} \quad (z, z^* \mapsto z_t, z_t^*)$$

is the injective algebra homomorphism from Proposition 1.1. Moreover every finite-dimensional irreducible representation of \mathcal{A} that belongs to $\mathcal{Irr}_d^{(b,b^*)}(\mathcal{A})$ arises in this way. In other words, a finite-dimensional irreducible representation $\rho' : \mathcal{A} \rightarrow \text{End}(V)$ that belongs to $\mathcal{Irr}_d^{(b,b^*)}(\mathcal{A})$ can be ‘extended’ via ι_t to a finite-dimensional irreducible representation $\rho : \mathcal{T} \rightarrow \text{End}(V)$ that belongs to $\mathcal{Irr}_d^{s,t}(\mathcal{T})$. Thus we have

Corollary 1.6 *If $b = st$, $b^* = st^{-1}$, then the mapping $\rho \mapsto \rho \circ \iota_t$ gives a bijection from $\mathcal{Irr}_d^{s,t}(\mathcal{T})$ to $\mathcal{Irr}_d^{(b,b^*)}(\mathcal{A})$, where $\iota_t : \mathcal{A} \rightarrow \mathcal{T}$ ($z, z^* \mapsto z_t, z_t^*$) is the injective algebra homomorphism from Proposition 1.1.*

Since $\mathcal{Irr}(\mathcal{A})$ is the disjoint union of the trivial representations and $\mathcal{Irr}_d^{(b,b^*)}(\mathcal{A})$ over $d \in \mathbb{N}$ and $(b, b^*) \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) / \sim$, Problem 1 is reduced to Problem 2 by Corollary 1.6. Namely, $\mathcal{Irr}(\mathcal{A})$ is the disjoint union of the trivial representations and

$$\{\rho \circ \iota_t \mid \rho \in \mathcal{Irr}_d^{s,t}(\mathcal{T})\} \quad (10)$$

over $d \in \mathbb{N}$ and $(s, t) \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) / \approx$, where the equivalence relation \approx is defined by $(s, t) \approx (s', t')$ if and only if

$$(s', t') \in \{\pm(s, t), \pm(t^{-1}, s^{-1}), \pm(t, s), \pm(s^{-1}, t^{-1})\} \quad \text{for the case } (\varepsilon, \varepsilon^*) = (1, 1), \quad (11)$$

$$(s', t') \in \{\pm(s, t), \pm(t^{-1}, s^{-1})\} \quad \text{for the case } (\varepsilon, \varepsilon^*) = (1, 0), \quad (12)$$

and $(s', t') = \pm(s, t)$ for the case $(\varepsilon, \varepsilon^*) = (0, 0)$.

As we see in the next proposition, the property (C_1) for $\mathcal{Irr}(\mathcal{A})$ is automatically satisfied when $(\varepsilon, \varepsilon^*) = (1, 1)$.

Proposition 1.7 *If $(\varepsilon, \varepsilon^*) = (1, 1)$, then every finite-dimensional irreducible representation $\rho : \mathcal{A} \longrightarrow \text{End}(V)$ satisfies the property (C_1) , i.e., $\rho(z), \rho(z^*)$ are diagonalizable.*

Proof. Regard V as an irreducible \mathcal{A} -module via ρ . For $\theta \in \mathbb{C}$, set $V(\theta) = \{v \in V | zv = \theta v\}$. Note that θ is an eigenvalue of $\rho(z)$ if and only if $V(\theta) \neq 0$, and in this case $V(\theta)$ is the corresponding eigenspace. Using the relation $[z, z^2 z^* - \beta z z^* z + z^* z^2] = \delta [z, z^*]$ with $\beta = q^2 + q^{-2}$, $\delta = -(q^2 - q^{-2})$, we find $(z - \theta^-)(z - \theta)(z - \theta^+)z^*v = 0$ for all $v \in V(\theta)$, where $\theta = \zeta + \zeta^{-1}$, $\theta^+ = q^2 \zeta + q^{-2} \zeta^{-1}$, $\theta^- = q^{-2} \zeta + q^2 \zeta^{-1}$, i.e.,

$$z^*V(\theta) \subseteq V(\theta^-) + V(\theta) + V(\theta^+).$$

Set $\theta^{(i)} = q^{2i} \zeta + q^{-2i} \zeta^{-1}$. Then $\sum_{i \in \mathbb{Z}} V(\theta^{(i)})$, which is a finite sum by $\dim V < \infty$, is invariant under z, z^* . Since z, z^* generate \mathcal{A} and V is irreducible as an \mathcal{A} -module, we have $V = \sum_{i \in \mathbb{Z}} V(\theta^{(i)})$. This implies that $\rho(z)$ is diagonalizable. Similarly, $\rho(z^*)$ is shown to be diagonalizable. \square

So for the case $(\varepsilon, \varepsilon^*) = (1, 1)$, Problem 1 is equivalent to

Problem 3 *Classify up to isomorphism the finite-dimensional irreducible representations of the q -Onsager algebra \mathcal{A}_1 .*

Thus the classification of standardized TD-pairs of Type I that are the 1st kind is equivalent to that of finite-dimensional irreducible representations of the q -Onsager algebra \mathcal{A}_1 .

1.3 Finite-dimensional irreducible \mathcal{T} -modules

Let $\rho : \mathcal{T} \longrightarrow \text{End}(V)$ be a finite-dimensional irreducible representation of the augmented TD-algebra \mathcal{T} . We regard V as an irreducible \mathcal{T} -module via ρ . Let us recall Lemma 1.2 in Section 1.2. The action of k on V is diagonalizable with eigenvalues $\{sq^{2i-d} | 0 \leq i \leq d\}$ for some nonzero $s \in \mathbb{C}$ and an integer $d \geq 0$. The scalar s and the integer d are called the type and the diameter, respectively. Let $V = \bigoplus_{i=0}^d U_i$ denote the eigenspace decomposition of the action of k on V , where U_i is the eigenspace belonging to sq^{2i-d} . It holds that $xU_i \subseteq U_{i+1}$, $yU_i \subseteq U_{i-1}$ ($0 \leq i \leq d$), where $U_{-1} = U_{d+1} = 0$. We call the direct sum $V = \bigoplus_{i=0}^d U_i$ the weight space decomposition and U_0 the highest weight space.

Theorem 1.8 *Let V be a finite-dimensional irreducible \mathcal{T} -module and $V = \bigoplus_{i=0}^d U_i$ the weight space decomposition. Then*

$$\dim U_i \leq \binom{d}{i} \quad (0 \leq i \leq d).$$

In particular U_0 has dimension 1.

Theorem 1.8 will be proved in Section 3. Since $xU_j \subseteq U_{j+1}$, $yU_j \subseteq U_{j-1}$ ($0 \leq j \leq d$) by Lemma 1.2, the highest weight space U_0 is invariant under $y^i x^i$ for every integer $i \geq 0$. Since $\dim U_0 = 1$ by Theorem 1.8, there exists $\sigma_i = \sigma_i(V) \in \mathbb{C}$ such that

$$y^i x^i v = \sigma_i v \quad (v \in U_0)$$

for every integer $i \geq 0$. Observe $\sigma_0 = 1$ and $\sigma_i = 0$ if $i > d$, where d is the diameter of the \mathcal{T} -module V . It is shown later that $\sigma_d \neq 0$. Let $\mathcal{M}_d^s(\mathcal{T})$ denote the set of isomorphism classes of finite-dimensional irreducible \mathcal{T} -modules with type s , diameter d , and Σ_d the set of sequences $\{\sigma_i\}_{i=0}^d$ of scalars $\sigma_i \in \mathbb{C}$ with $\sigma_0 = 1$, $\sigma_d \neq 0$. Then we have a mapping σ from $\mathcal{M}_d^s(\mathcal{T})$ to Σ_d that sends V to $\{\sigma_i(V)\}_{i=0}^d$, where $\sigma_i(V)$ is the eigenvalue of $y^i x^i$ on the highest weight space of V .

Theorem 1.9 *For each nonzero $s \in \mathbb{C}$, the mapping*

$$\sigma : \mathcal{M}_d^s(\mathcal{T}) \longrightarrow \Sigma_d \quad (V \longmapsto \{\sigma_i(V)\}_{i=0}^d)$$

is a bijection.

The fact $\sigma_d(V) \neq 0$ and the injectiveness of σ will be proved in Section 3. The surjectiveness of σ will be proved in Section 5.

For a finite-dimensional irreducible \mathcal{T} -module V of type s and diameter d , we define a monic polynomial $P_V(\lambda)$ of degree d in λ as follows:

$$P_V(\lambda) = Q^{-1} \sum_{i=0}^d \sigma_i(V) \prod_{j=i+1}^d (q^j - q^{-j})^2 (\varepsilon s^{-2} q^{2(d-j)} + \varepsilon^* s^2 q^{-2(d-j)} - \lambda),$$

where $\sigma_i(V)$ is the eigenvalue of $y^i x^i$ on the highest weight space of V and

$$Q = Q_d = (-1)^d (q - q^{-1})^2 (q^2 - q^{-2})^2 \cdots (q^d - q^{-d})^2.$$

The polynomial $P_V(\lambda)$ is called the *Drinfel'd polynomial* of the \mathcal{T} -module V . Note that the parameters q and $(\varepsilon, \varepsilon^*)$ in the definition of $P_V(\lambda)$ are independent of the \mathcal{T} -module V , since they are chosen and fixed in advance for the augmented TD-algebra \mathcal{T} .

Remark 1.10 The following identities directly follow from the definition of $P_V(\lambda)$.

(i) For $\lambda = \varepsilon s^{-2} + \varepsilon^* s^2$,

$$P_V(\lambda) = Q^{-1} \sigma_d(V).$$

(ii) For $\lambda = t^2 + \varepsilon \varepsilon^* t^{-2}$ with t an arbitrary nonzero scalar,

$$P_V(\lambda) = Q^{-1} \sum_{i=0}^d \sigma_i(V) (\theta_0 - \theta_{i+1}) \cdots (\theta_0 - \theta_d) (\theta_0^* - \theta_{i+1}^*) \cdots (\theta_0^* - \theta_d^*),$$

where $\theta_i = s t q^{2i-d} + \varepsilon s^{-1} t^{-1} q^{d-2i}$, $\theta_i^* = \varepsilon^* s t^{-1} q^{2i-d} + s^{-1} t q^{d-2i}$.

Let \mathcal{P}_d^s denote the set of monic polynomials $P(\lambda)$ of degree d in λ such that

$$P(\lambda) \neq 0 \quad \text{for } \lambda = \varepsilon s^{-2} + \varepsilon^* s^2.$$

Then the mapping that sends $\{\sigma_i\}_{i=0}^d$ to

$$P(\lambda) = Q^{-1} \sum_{i=0}^d \sigma_i \prod_{j=i+1}^d (q^j - q^{-j})^2 (\varepsilon s^{-2} q^{2(d-j)} + \varepsilon^* s^2 q^{-2(d-j)} - \lambda)$$

gives a bijection from Σ_d to \mathcal{P}_d^s . So we can restate Theorem 1.9 as follows.

Theorem 1.9' *The mapping $V \mapsto P_V(\lambda)$ gives a bijection from $\mathcal{M}_d^s(\mathcal{T})$ to \mathcal{P}_d^s .*

This gives a parametrization of the set $\mathcal{M}_d^s(\mathcal{T})$ in question in Problem 2 (i).

Theorem 1.11 *Let V be a finite-dimensional irreducible \mathcal{T} -module of type s and diameter d . Assume that the property $(C_1)_t$ holds for some $t \in \mathbb{C}$, i.e., the actions of z_t, z_t^* on V are both diagonalizable. Then V is irreducible as a $\langle z_t, z_t^* \rangle$ -module if and only if $P_V(\lambda) \neq 0$ for $\lambda = t^2 + \varepsilon \varepsilon^* t^{-2}$. Here $P_V(\lambda)$ is the Drinfel'd polynomial of the \mathcal{T} -module V .*

Theorem 1.11 will be proved in Section 4. Theorem 1.11 together with Proposition 1.3 gives a parametrization of the representations of \mathcal{T} in question in Problem 2 (ii). For an integer $d \geq 1$ and nonzero $s, t \in \mathbb{C}$, define the sets $\mathcal{M}_d^{s,t}(\mathcal{T})$ and $\mathcal{P}_d^{s,t}$ as follows. $\mathcal{M}_d^{s,t}(\mathcal{T})$ denotes the set of isomorphism classes of finite-dimensional irreducible \mathcal{T} -modules V with type s , diameter d that satisfy the properties $(C_1)_t, (C_2)_t$, i.e., the actions of z_t, z_t^* on V are both diagonalizable and V is irreducible as a $\langle z_t, z_t^* \rangle$ -module. $\mathcal{P}_d^{s,t}$ denotes the set of monic polynomials $P(\lambda)$ of degree d in λ such that $P(\lambda) \neq 0$ for $\lambda = \varepsilon s^{-2} + \varepsilon^* s^2$ and $\lambda = t^2 + \varepsilon \varepsilon^* t^{-2}$. Note that $\mathcal{M}_d^{s,t}(\mathcal{T})$ (resp. $\mathcal{P}_d^{s,t}(\mathcal{T})$) is a subset of $\mathcal{M}_d^s(\mathcal{T})$ (resp. $\mathcal{P}_d^s(\mathcal{T})$) and $\mathcal{M}_d^s(\mathcal{T})$ is bijectively mapped to \mathcal{P}_d^s by $V \mapsto P_V(\lambda)$ by Theorem 1.9'. Let V be a finite-dimensional irreducible \mathcal{T} -module that belongs to $\mathcal{M}_d^s(\mathcal{T})$. Then by Proposition 1.3, the property $(C_1)_t$ holds for the \mathcal{T} -module V if and only if

$$st \neq \pm \varepsilon q^i \quad \text{for any integer } i \quad (1-d \leq i \leq d-1), \quad (13)$$

$$st^{-1} \neq \pm \varepsilon^* q^i \quad \text{for any integer } i \quad (1-d \leq i \leq d-1). \quad (14)$$

Thus if one of the conditions (13), (14) fails, then $\mathcal{M}_d^{s,t}(\mathcal{T})$ is empty. Suppose each of (13), (14) holds. Then by Theorem 1.11, the property $(C_2)_t$ holds for the \mathcal{T} -module V if and only if $P_V(\lambda) \neq 0$ for $\lambda = t^2 + \varepsilon \varepsilon^* t^{-2}$. So $\mathcal{M}_d^{s,t}(\mathcal{T})$ is precisely mapped onto $\mathcal{P}_d^{s,t}$ by the bijection $V \mapsto P_V(\lambda)$ from $\mathcal{M}_d^s(\mathcal{T})$ to \mathcal{P}_d^s . Thus we have

Corollary 1.12 *If one of the conditions (13), (14) fails, then $\mathcal{M}_d^{s,t}(\mathcal{T})$ is empty. Suppose each of (13), (14) holds. Then the mapping $V \mapsto P_V(\lambda)$ gives a bijection from $\mathcal{M}_d^{s,t}(\mathcal{T})$ to $\mathcal{P}_d^{s,t}$.*

This gives a parametrization of the set $\mathcal{M}_d^{s,t}(\mathcal{T})$ in question in Problem 2 (ii). Since $\mathcal{M}_d^{s,t}(\mathcal{T})$ can be naturally identified with $\text{Irr}_d^{s,t}(\mathcal{T})$, Corollary 1.12 gives a parametrization of $\mathcal{STD}_d^{(b,b^*)}$ through Proposition 1.5, where $b = st, b^* = st^{-1}$.

1.4 Construction of finite-dimensional irreducible \mathcal{T} -modules

Given $(\varepsilon, \varepsilon^*) \in \{(1, 1), (1, 0), (0, 0)\}$ and a nonzero scalar q that is not a root of unity, let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$ denote the augmented TD-algebra. \mathcal{T} is generated by $x, y, k^{\pm 1}$ subject to the relations $(\text{TD})'_0, (\text{TD})'$ in Section 1.2. In the next proposition, we give an injective algebra-homomorphism φ_s of \mathcal{T} into the $U_q(\mathfrak{sl}_2)$ -loop algebra $\mathcal{L} = U_q(L(\mathfrak{sl}_2))$ for each nonzero scalar $s \in \mathbb{C}$. \mathcal{L} is the associative \mathbb{C} -algebra with 1 defined by generators $e_i^+, e_i^-, k_i, k_i^{-1}$ ($i = 0, 1$) subject to the relations

$$\begin{aligned} k_0 k_1 &= k_1 k_0 = 1, \\ k_i k_i^{-1} &= k_i^{-1} k_i = 1, \\ k_i e_i^{\pm} k_i^{-1} &= q^{\pm 2} e_i^{\pm}, \\ k_i e_j^{\pm} k_i^{-1} &= q^{\mp 2} e_j^{\pm} \quad (i \neq j), \\ [e_i^+, e_i^-] &= \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\ [e_i^+, e_j^-] &= 0 \quad (i \neq j), \\ [e_i^{\pm}, (e_i^{\pm})^2 e_j^{\pm} - (q^2 + q^{-2}) e_i^{\pm} e_j^{\pm} e_i^{\pm} + e_j^{\pm} (e_i^{\pm})^2] &= 0 \quad (i \neq j). \end{aligned}$$

Note that if we replace $k_0 k_1 = k_1 k_0 = 1$ in the defining relations for \mathcal{L} by $k_0 k_1 = k_1 k_0$, then we have the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$: \mathcal{L} is isomorphic to the quotient algebra of $U_q(\widehat{\mathfrak{sl}}_2)$ by the two-sided ideal generated by $k_0 k_1 - 1$.

Proposition 1.13 *For each nonzero $s \in \mathbb{C}$, there exists an algebra homomorphism φ_s from \mathcal{T} to \mathcal{L} that sends x, y, k to $x(s), y(s), k(s)$, respectively, where*

$$\begin{aligned} x(s) &= \alpha (s e_0^+ + \varepsilon s^{-1} e_1^- k_1) \quad \text{with } \alpha = -q^{-1}(q - q^{-1})^2, \\ y(s) &= \varepsilon^* s e_0^- k_0 + s^{-1} e_1^+, \\ k(s) &= s k_0. \end{aligned}$$

Moreover φ_s is injective.

The existence of φ_s follows from the fact that the relations $(\text{TD})'_0, (\text{TD})'$ hold for $x(s), y(s), k(s), k(s)^{-1}$. We leave the tedious calculations of checking it to the reader. The injectivity of φ_s will be proved in Section 2.

Let \mathcal{L}' denote the subalgebra of \mathcal{L} generated by $e_0^+, e_1^{\pm}, k_i^{\pm 1}$ ($i = 0, 1$): e_0^- is missing from the set of generators for \mathcal{L}' . Let \mathcal{B} denote the subalgebra of \mathcal{L} generated by $e_0^+, e_1^+, k_i^{\pm 1}$ ($i = 0, 1$), the Borel subalgebra of \mathcal{L} . Observe $\mathcal{B} \subseteq \mathcal{L}'$. Note that the image of φ_s is contained in \mathcal{L}' if $(\varepsilon, \varepsilon^*) = (1, 0)$ and it coincides with \mathcal{B} if $(\varepsilon, \varepsilon^*) = (0, 0)$. If $(\varepsilon, \varepsilon^*) = (1, 1)$ (resp. $(1, 0), (0, 0)$), each finite-dimensional irreducible \mathcal{L} -module (resp. \mathcal{L}' -module, \mathcal{B} -module) can be regarded as a \mathcal{T} -module via the injective algebra homomorphism $\varphi_s : \mathcal{T} \rightarrow \mathcal{L}$. Such a \mathcal{T} -module is called a \mathcal{T} -module via φ_s . We determine when a finite-dimensional irreducible \mathcal{L} -module (resp. \mathcal{L}' -module, \mathcal{B} -module) remains irreducible as a \mathcal{T} -module via φ_s , and by finding an explicit formula for the Drinfel'd polynomial $P_V(\lambda)$, we show that every finite-dimensional irreducible \mathcal{T} -module with type s arises in this way via φ_s (see Theorem 1.9').

We give an overview of finite-dimensional representations of \mathcal{L} that we need to state our explicit construction of irreducible \mathcal{T} -modules via φ_s . For $a \in \mathbb{C}$ ($a \neq 0$) and $\ell \in \mathbb{Z}$ ($\ell > 0$), $V(\ell, a)$ denotes the *evaluation module* of \mathcal{L} , i.e., $V(\ell, a)$ is an $(\ell+1)$ -dimensional vector space over \mathbb{C} with a basis v_0, v_1, \dots, v_ℓ on which \mathcal{L} acts as follows:

$$\begin{aligned} k_0 v_i &= q^{2i-\ell} v_i, \\ k_1 v_i &= q^{\ell-2i} v_i, \\ e_0^+ v_i &= a q [i+1] v_{i+1}, \\ e_0^- v_i &= a^{-1} q^{-1} [\ell-i+1] v_{i-1}, \\ e_1^+ v_i &= [\ell-i+1] v_{i-1}, \\ e_1^- v_i &= [i+1] v_{i+1}, \end{aligned}$$

where $v_{-1} = v_{\ell+1} = 0$ and $[j] = [j]_q = (q^j - q^{-j})/(q - q^{-1})$. $V(\ell, a)$ is an irreducible \mathcal{L} -module. We call v_0, v_1, \dots, v_ℓ a *standard basis*.

Let Δ denote the *coproduct* of \mathcal{L} : the algebra homomorphism from \mathcal{L} to $\mathcal{L} \otimes \mathcal{L}$ defined by

$$\begin{aligned} \Delta(k_i^{\pm 1}) &= k_i^{\pm 1} \otimes k_i^{\pm 1}, \\ \Delta(e_i^+) &= k_i \otimes e_i^+ + e_i^+ \otimes 1, \\ \Delta(e_i^- k_i) &= k_i \otimes e_i^- k_i + e_i^- k_i \otimes 1. \end{aligned}$$

Given \mathcal{L} -modules V_1, V_2 , the tensor product $V_1 \otimes V_2$ becomes an \mathcal{L} -module via Δ . Given a set of evaluation modules $V(\ell_i, a_i)$ ($1 \leq i \leq n$) of \mathcal{L} , the tensor product

$$V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

makes sense as an \mathcal{L} -module without being affected by the parentheses for the tensor product because of the coassociativity of Δ .

With an evaluation module $V(\ell, a)$ of \mathcal{L} , we associate the set $S(\ell, a)$ of scalars $a q^{-\ell+1}, a q^{-\ell+3}, \dots, a q^{\ell-1}$:

$$S(\ell, a) = \{a q^{2i-\ell+1} \mid 0 \leq i \leq \ell-1\}.$$

The set $S(\ell, a)$ is called a *q-string* of length ℓ . Two *q-strings* $S(\ell, a), S(\ell', a')$ are said to be *adjacent* if $S(\ell, a) \cup S(\ell', a')$ is a longer *q-string*, i.e., $S(\ell, a) \cup S(\ell', a') = S(\ell'', a'')$ for some ℓ'', a'' with $\ell'' > \max\{\ell, \ell'\}$. It can be easily checked that $S(\ell, a), S(\ell', a')$ are adjacent if and only if $a^{-1}a' = q^{\pm i}$ for some

$$i \in \{|\ell - \ell'| + 2, |\ell - \ell'| + 4, \dots, \ell + \ell'\}.$$

Two *q-strings* $S(\ell, a), S(\ell', a')$ are defined to be *in general position* if they are not adjacent, i.e., if either

- (i) $S(\ell, a) \cup S(\ell', a')$ is not a *q-string*,
- or
- (ii) $S(\ell, a) \subseteq S(\ell', a')$ or $S(\ell, a) \supseteq S(\ell', a')$.

A multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is said to be *in general position* if $S(\ell_i, a_i)$ and $S(\ell_j, a_j)$ are in general position for any i, j ($i \neq j$, $1 \leq i \leq n$, $1 \leq j \leq n$). The following fact is well-known and easy to prove. Let Ω be a finite multi-set of nonzero scalars from \mathbb{C} . Then there exists a multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings in general position such that

$$\Omega = \bigcup_{i=1}^n S(\ell_i, a_i)$$

as multi-sets of nonzero scalars. Moreover such a multi-set of q -strings is uniquely determined by Ω .

With a tensor product $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules $V(\ell_i, a_i)$ ($1 \leq i \leq n$), we associate the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings. The following (i), (ii), (iii) are well-known [2]:

(i) A tensor product $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules is irreducible as an \mathcal{L} -module if and only if the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is in general position.

(ii) Set $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_{n'}, a'_{n'})$ and assume that V, V' are both irreducible as an \mathcal{L} -module. Then V, V' are isomorphic as \mathcal{L} -modules if and only if the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ coincide, i.e., $n = n'$ and $\ell_i = \ell'_i, a_i = a'_i$ for all i ($1 \leq i \leq n$) with a suitable reordering of $S(\ell'_1, a'_1), \dots, S(\ell'_{n'}, a'_{n'})$.

(iii) Every nontrivial finite-dimensional irreducible \mathcal{L} -module of type (1,1) is isomorphic to some $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$.

Two multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ of q -strings are defined to be *equivalent* if there exists $\varepsilon_i \in \{\pm 1\}$ ($1 \leq i \leq n$) such that $\{S(\ell_i, a_i^{\varepsilon_i})\}_{i=1}^n$ and $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ coincide, i.e., $n = n'$ and $\ell_i = \ell'_i, a_i^{\varepsilon_i} = a'_i$ for all i ($1 \leq i \leq n$) with a suitable reordering of $S(\ell'_1, a'_1), \dots, S(\ell'_n, a'_n)$. A multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is defined to be *strongly in general position* if any multi-set of q -strings equivalent to $\{S(\ell_i, a_i)\}_{i=1}^n$ is in general position, i.e., the multi-set $\{S(\ell_i, a_i^{\varepsilon_i})\}_{i=1}^n$ is in general position for any choice of $\varepsilon_i \in \{\pm 1\}$ ($1 \leq i \leq n$).

Lemma 1.14 *Let Ω be a finite multi-set of nonzero scalars from \mathbb{C} such that c and c^{-1} appear in Ω in pairs, i.e., c and c^{-1} have the same multiplicity in Ω for each $c \in \Omega$, where we understand that if 1 or -1 appears in Ω , it has even multiplicity. Then there exists a multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings strongly in general position such that*

$$\Omega = \bigcup_{i=1}^n (S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1}))$$

as multi-sets of nonzero scalars. Such a multi-set of q -strings is uniquely determined by Ω up to equivalence.

Lemma 1.14 will be proved in Section 7.

Theorem 1.15 (Case $(\varepsilon, \varepsilon^*) = (1, 1)$) *Let $\mathcal{T} = \mathcal{T}_q^{(1,1)}$ denote the augmented TD-algebra of the 1st kind. The following (i), (ii), (iii) hold.*

- (i) *A tensor product $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules is irreducible as a \mathcal{T} -module via φ_s if and only if $-s^2 \notin S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})$ for all i ($1 \leq i \leq n$) and the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is strongly in general position. In this case, the \mathcal{T} -module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s has type s and diameter $d = \ell_1 + \cdots + \ell_n$ and the Drinfel'd polynomial $P_V(\lambda)$ of the \mathcal{T} -module V via φ_s is*

$$P_V(\lambda) = \prod_{i=1}^n P_{V(\ell_i, a_i)}(\lambda),$$

where

$$P_{V(\ell_i, a_i)}(\lambda) = \prod_{c \in S(\ell_i, a_i)} (\lambda + c + c^{-1}).$$

- (ii) *Set $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_n, a'_n)$ and assume that V, V' are both irreducible as a \mathcal{T} -module via φ_s . Then V, V' are isomorphic as \mathcal{T} -modules via φ_s if and only if the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ of q -strings are equivalent.*
- (iii) *Every nontrivial finite-dimensional irreducible \mathcal{T} -module of type s is isomorphic to some \mathcal{T} -module $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s .*

Theorem 1.15 will be proved in Section 7. Note that the Drinfel'd polynomial of an irreducible \mathcal{T} -module $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s is determined by the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings and independent of φ_s . Problem 2, which is to determine $\mathcal{M}_d^s(\mathcal{T})$ and $\mathcal{M}_d^{s,t}(\mathcal{T})$, is solved by Theorem 1.15 as follows in the case of $(\varepsilon, \varepsilon^*) = (1, 1)$. Assume $(\varepsilon, \varepsilon^*) = (1, 1)$. The set $\mathcal{M}_d^s(\mathcal{T})$ is determined in terms of tensor products of evaluation modules by Theorem 1.15 (i), (ii), (iii). Recall the bijection $V \mapsto P_V(\lambda)$ from $\mathcal{M}_d^s(\mathcal{T})$ to \mathcal{P}_d^s in Theorem 1.9'. The subset $\mathcal{M}_d^{s,t}(\mathcal{T})$ of $\mathcal{M}_d^s(\mathcal{T})$ is nonempty if and only if the conditions (13), (14) hold, i.e.,

$$\pm st, \pm st^{-1} \notin \{q^i \mid i = -d + 1, -d + 2, \dots, d - 1\}, \quad (15)$$

and in this case $\mathcal{M}_d^{s,t}(\mathcal{T})$ is mapped onto $\mathcal{P}_d^{s,t}$ by the bijection $V \mapsto P_V(\lambda)$ (see Corollary 1.12). For an irreducible \mathcal{T} -module $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s , we find by Theorem 1.15 (i) that $P_V(\lambda)$ does not vanish at $\lambda = t^2 + t^{-2}$ if and only if $-t^2 \notin S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})$ for all i ($1 \leq i \leq n$). Thus we have

Corollary 1.16 *Assume $(\varepsilon, \varepsilon^*) = (1, 1)$. Then $\mathcal{M}_d^s(\mathcal{T})$ and $\mathcal{M}_d^{s,t}(\mathcal{T})$ are determined as follows.*

- (i) *$\mathcal{M}_d^s(\mathcal{T})$ consists of the isomorphism classes of \mathcal{T} -modules $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s with the properties that*

- (i.1) the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is strongly in general position,
- (i.2) $-s^2 \notin S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})$ for all i ($1 \leq i \leq n$),
- (i.3) $d = \ell_1 + \cdots + \ell_n$.

The isomorphism classes of such \mathcal{T} -modules $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s are in one-to-one correspondence with the equivalence classes of the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings that have the properties (i.1), (i.2), (i.3) above.

- (ii) $\mathcal{M}_d^{s,t}(\mathcal{T})$ is nonempty if and only if the condition (15) holds. Suppose the condition (15) holds. Then $\mathcal{M}_d^{s,t}(\mathcal{T})$ consists of the isomorphism classes of \mathcal{T} -modules $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s with the properties (i.1), (i.2), (i.3) above and
 - (ii.1) $-t^2 \notin S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})$ for all i ($1 \leq i \leq n$).

The next theorem follows from Corollary 1.16 and [8, Proposition 7.15]. It solves Problem 3, which is to determine the finite-dimensional irreducible representations of the q -Onsager algebra up to isomorphism. For an \mathcal{L} -module V , let ρ_V denote the representation of \mathcal{L} afforded by the \mathcal{L} -module V . Then $\rho_V \circ \varphi_s$ is the representation of \mathcal{T} afforded by the \mathcal{T} -module V via φ_s , and $\rho_V \circ \varphi_s \circ \iota_t$ is a representation of \mathcal{A} , where

$$\iota_t : \mathcal{A} \longrightarrow \mathcal{T} \quad (z, z^* \mapsto z_t, z_t^*)$$

is the injective algebra homomorphism from Proposition 1.1.

Theorem 1.17 Assume $(\varepsilon, \varepsilon^*) = (1, 1)$. Let $\mathcal{A} = \mathcal{A}_q^{(1,1)}$ denote the q -Onsager algebra. The following (i), (ii), (iii) hold.

- (i) For an \mathcal{L} -module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ and nonzero $s, t \in \mathbb{C}$, the representation $\rho_V \circ \varphi_s \circ \iota_t$ of \mathcal{A} is irreducible if and only if
 - (i.1) the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is strongly in general position,
 - (i.2) none of $-s^2, -t^2$ belongs to $S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})$ for any i ($1 \leq i \leq n$),
 - (i.3) none of the four scalars $\pm st, \pm st^{-1}$ equals q^i for any $i \in \mathbb{Z}$ ($-d+1 \leq i \leq d-1$), where $d = \ell_1 + \cdots + \ell_n$.
- (ii) For \mathcal{L} -modules $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_n, a'_n)$ and $(s, t), (s', t') \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$, set $\rho = \rho_V \circ \varphi_s \circ \iota_t$ and $\rho' = \rho_{V'} \circ \varphi_{s'} \circ \iota_{t'}$. Assume that the representations ρ, ρ' of \mathcal{A} are both irreducible. Then they are isomorphic as representations of \mathcal{A} if and only if the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ are equivalent and $(s, t) \approx (s', t')$ in the sense of (11), i.e.,

$$(s', t') \in \{\pm(s, t), \pm(t^{-1}, s^{-1}), \pm(t, s), \pm(s^{-1}, t^{-1})\}.$$

- (iii) Every nontrivial finite-dimensional irreducible representation of \mathcal{A} is isomorphic to $\rho_V \circ \varphi_s \circ \iota_t$ for some \mathcal{L} -module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ and $(s, t) \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$.

Proof. The assertions (i), (iii) follow from Corollary 1.6 and Corollary 1.16, since $\mathcal{Irr}_d^{s,t}(\mathcal{T})$ is naturally identified with $\mathcal{M}_d^{s,t}(\mathcal{T})$. To prove the assertion (ii), suppose that the irreducible representations $\rho = \rho_V \circ \varphi_s \circ \iota_t$ and $\rho' = \rho_{V'} \circ \varphi_{s'} \circ \iota_{t'}$ of \mathcal{A} are isomorphic, where $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_{n'}, a'_{n'})$. Set $A = \rho(z)$, $A^* = \rho(z^*)$ and $B = \rho'(z)$, $B^* = \rho'(z^*)$. Then A, A^* are a TD-pair belonging to $\mathcal{STD}_d^{(b,b^*)}$, where $b = st$, $b^* = st^{-1}$, $d = \ell_1 + \cdots + \ell_n$, and B, B^* are a TD-pair belonging to $\mathcal{STD}_{d'}^{(c,c^*)}$, where $c = s't'$, $c^* = s't'^{-1}$, $d' = \ell'_1 + \cdots + \ell'_{n'}$ (see Proposition 1.5). Since ρ, ρ' are isomorphic, the TD-pair A, A^* is isomorphic to the TD-pair B, B^* , so we have $(b, b^*) \sim (c, c^*)$ in the sense of (8), i.e., $(s, t) \approx (s', t')$ in the sense of (11). Moreover by [8, Proposition 7.15], the Drinfel'd polynomial $P_V(\lambda)$ of the \mathcal{T} -module V via φ_s coincides with the Drinfel'd polynomial $P_{V'}(\lambda)$ of the \mathcal{T} -module V' via $\varphi_{s'}$. By Theorem 1.15 (i) and Lemma 1.14, the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ of q -strings are equivalent.

Conversely for the irreducible representations $\rho = \rho_V \circ \varphi_s \circ \iota_t$ and $\rho' = \rho_{V'} \circ \varphi_{s'} \circ \iota_{t'}$ of \mathcal{A} with $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_{n'}, a'_{n'})$, suppose that $(s, t) \approx (s', t')$ and the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ of q -strings are equivalent. Set $b = st$, $b^* = st^{-1}$, $c = s't'$, $c^* = s't'^{-1}$ and $d = \ell_1 + \cdots + \ell_n$, $d' = \ell'_1 + \cdots + \ell'_{n'}$. Then $(b, b^*) \sim (c, c^*)$ and $d = d'$, so $\mathcal{STD}_d^{(b,b^*)} = \mathcal{STD}_{d'}^{(c,c^*)}$. Set $A = \rho(z)$, $A^* = \rho(z^*)$. Then A, A^* is a TD-pair belonging to $\mathcal{STD}_d^{(b,b^*)}$, so it belongs to $\mathcal{STD}_{d'}^{(c,c^*)}$: the difference is the orderings of the eigenspaces of A, A^* . Apply Proposition 1.5 to $\mathcal{STD}_{d'}^{(c,c^*)}$. Then there exists a unique representation ρ'' of \mathcal{T} up to isomorphism belonging to $\mathcal{Irr}_{d'}^{s',t'}(\mathcal{T})$ such that the TD-pair $B = \rho'' \circ \iota_{t'}(z)$, $B^* = \rho'' \circ \iota_{t'}(z^*)$ is isomorphic to A, A^* . By Theorem 1.15 (iii), we may assume $\rho'' = \rho_{V''} \circ \varphi_{s'} \circ \iota_{t'}$ for some $V'' = V(\ell''_1, a''_1) \otimes \cdots \otimes V(\ell''_{n''}, a''_{n''})$. Apparently, $\rho'' \circ \iota_{t'} = \rho_{V''} \circ \varphi_{s'} \circ \iota_{t'}$ is isomorphic to $\rho = \rho_V \circ \varphi_s \circ \iota_t$ as representations of \mathcal{A} , since the TD-pair $B = \rho'' \circ \iota_{t'}(z)$, $B^* = \rho'' \circ \iota_{t'}(z^*)$ is isomorphic to $A = \rho(z)$, $A^* = \rho(z^*)$. Then by what we have already proved in the 1st half of the proof, the multi-set $\{S(\ell''_i, a''_i)\}_{i=1}^{n''}$ of q -strings is equivalent to $\{S(\ell_i, a_i)\}_{i=1}^n$ and hence to $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$. This means $\rho'' \circ \iota_{t'} = \rho_{V''} \circ \varphi_{s'} \circ \iota_{t'}$ is isomorphic to $\rho' = \rho_{V'} \circ \varphi_{s'} \circ \iota_{t'}$ as representations of \mathcal{A} . So ρ, ρ' are isomorphic as representations of \mathcal{A} . This completes the proof of the theorem. \square

Next we consider the case $(\varepsilon, \varepsilon^*) = (1, 0)$. Then $\varphi_s(\mathcal{T}) \subseteq \mathcal{L}'$. Note that the subalgebra \mathcal{L}' of \mathcal{L} is, by the triangular decomposition of \mathcal{L} , isomorphic to the algebra generated by the symbols e_0^+, e_1^\pm, k_i^\pm ($i = 0, 1$) subject to the defining relations

$$\begin{aligned} k_0 k_1 &= k_1 k_0 = 1, \\ k_i k_i^{-1} &= k_i^{-1} k_i = 1, \\ k_0 e_0^+ k_0^{-1} &= q^2 e_0^+, \\ k_1 e_1^\pm k_1^{-1} &= q^{\pm 2} e_1^\pm, \\ [e_0^+, e_1^-] &= 0, \\ [e_1^+, e_1^-] &= \frac{k_1 - k_1^{-1}}{q - q^{-1}}, \\ [e_i^+, (e_i^+)^2 e_j^+ - (q^2 + q^{-2}) e_i^+ e_j^+ e_i^+ + e_j^+ (e_i^+)^2] &= 0 \quad (i \neq j). \end{aligned}$$

So the evaluation module $V(\ell, a)$ makes sense as an \mathcal{L}' -module even for $a = 0$: for the standard basis v_0, v_1, \dots, v_d of $V(\ell, a)$,

$$\begin{aligned} k_0 v_i &= q^{2i-\ell} v_i \\ k_1 v_i &= q^{\ell-2i} v_i, \\ e_0^+ v_i &= a q[i+1] v_{i+1}, \\ e_1^+ v_i &= [\ell-i+1] v_{i-1}, \\ e_1^- v_i &= [i+1] v_{i+1}. \end{aligned}$$

For a positive integer ℓ and a scalar $a \in \mathbb{C}$, allowing $a = 0$, $V(\ell, a)$ is irreducible as an \mathcal{L}' -module and called an *evaluation module for \mathcal{L}'* . Since the coproduct Δ of \mathcal{L} is closed for \mathcal{L}' , i.e., $\Delta(\mathcal{L}') \subseteq \mathcal{L}' \otimes \mathcal{L}'$, the tensor product $V(\ell_1, a_1) \otimes \dots \otimes V(\ell_n, a_n)$ of evaluation modules for \mathcal{L}' becomes an \mathcal{L}' -module. We denote by $V(\ell)$ the evaluation module $V(\ell, 0)$. We allow $\ell = 0$ for $V(\ell)$ and understand that $V(0)$ is the trivial \mathcal{L}' -module, i.e., the 1-dimensional space on which $k_i^{\pm 1} = 1$, the identity map, $e_0^+ = e_1^{\pm} = 0$, the zero map. Thus $V(\ell, a)$ means the evaluation module for \mathcal{L}' with $\ell \geq 1, a \neq 0$ and $V(\ell)$ the evaluation module $V(\ell, 0)$ for \mathcal{L}' with $\ell \geq 0$.

Theorem 1.18 (Case $(\varepsilon, \varepsilon^*) = (1, 0)$) *Let $\mathcal{T} = \mathcal{T}_q^{(1,0)}$ denote the augmented TD-algebra of the 2nd kind. The following (i), (ii), (iii) hold.*

- (i) *A tensor product $V(\ell) \otimes V(\ell_1, a_1) \otimes \dots \otimes V(\ell_n, a_n)$ of evaluation modules for \mathcal{L}' is irreducible as a \mathcal{T} -module via φ_s if and only if $-s^{-2} \notin S(\ell_i, a_i)$ for all i ($1 \leq i \leq n$) and the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is in general position. In this case, the \mathcal{T} -module $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \dots \otimes V(\ell_n, a_n)$ via φ_s has type s and diameter $d = \ell + \ell_1 + \dots + \ell_n$ and the Drinfel'd polynomial $P_V(\lambda)$ of the \mathcal{T} -module V via φ_s is*

$$P_V(\lambda) = P_{V(\ell)}(\lambda) \prod_{i=1}^n P_{V(\ell_i, a_i)}(\lambda),$$

where

$$\begin{aligned} P_{V(\ell)}(\lambda) &= \lambda^\ell, \\ P_{V(\ell_i, a_i)}(\lambda) &= \prod_{c \in S(\ell_i, a_i)} (\lambda + c). \end{aligned}$$

- (ii) *Set $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \dots \otimes V(\ell_n, a_n)$, $V' = V(\ell') \otimes V(\ell'_1, a'_1) \otimes \dots \otimes V(\ell'_n, a'_n)$ and assume that V, V' are both irreducible as a \mathcal{T} -module via φ_s . Then V, V' are isomorphic as \mathcal{T} -modules via φ_s if and only if $\ell = \ell'$ and the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ of q -strings coincide, i.e., $n = n', \ell_i = \ell'_i, a_i = a'_i$ for all i ($0 \leq i \leq n$) with a suitable reordering of $S(\ell'_1, a'_1), \dots, S(\ell'_n, a'_n)$.*
- (iii) *Every nontrivial finite-dimensional irreducible \mathcal{T} -module of type s is isomorphic to some \mathcal{T} -module $V(\ell) \otimes V(\ell_1, a_1) \otimes \dots \otimes V(\ell_n, a_n)$ via φ_s .*

Theorem 1.18 will be proved in Section 7. Note that the Drinfel'd polynomial of an irreducible \mathcal{T} -module $V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s is determined by ℓ and the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings, independent of φ_s . Problem 2, which is to determine $\mathcal{M}_d^s(\mathcal{T})$ and $\mathcal{M}_d^{s,t}(\mathcal{T})$, is solved by Theorem 1.18 as follows in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$. Assume $(\varepsilon, \varepsilon^*) = (1, 0)$. The set $\mathcal{M}_d^s(\mathcal{T})$ is determined in terms of tensor products of evaluation modules by Theorem 1.18 (i), (ii), (iii). Recall the bijection $V \mapsto P_V(\lambda)$ from $\mathcal{M}_d^s(\mathcal{T})$ to \mathcal{P}_d^s in Theorem 1.9'. The subset $\mathcal{M}_d^{s,t}(\mathcal{T})$ of $\mathcal{M}_d^s(\mathcal{T})$ is nonempty if and only if the conditions (13), (14) hold, i.e.,

$$\pm st \notin \{q^i \mid i = -d+1, -d+2, \dots, d-1\}, \quad (16)$$

and in this case $\mathcal{M}_d^{s,t}(\mathcal{T})$ is mapped onto $\mathcal{P}_d^{s,t}$ by the bijection $V \mapsto P_V(\lambda)$ (see Corollary 1.12). For an irreducible \mathcal{T} -module $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s , we find by Theorem 1.18 (i) that $P_V(\lambda)$ does not vanish at $\lambda = t^2$ if and only if $-t^2 \notin S(\ell_i, a_i)$ for all i ($1 \leq i \leq n$). Thus we have

Corollary 1.19 *Assume $(\varepsilon, \varepsilon^*) = (1, 0)$. Then $\mathcal{M}_d^s(\mathcal{T})$ and $\mathcal{M}_d^{s,t}(\mathcal{T})$ are determined as follows.*

(i) $\mathcal{M}_d^s(\mathcal{T})$ consists of the isomorphism classes of \mathcal{T} -modules $V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s with the properties that

(i.1) the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is in general position,

(i.2) $-s^{-2} \notin S(\ell_i, a_i)$ for all i ($1 \leq i \leq n$),

(i.3) $d = \ell + \ell_1 + \cdots + \ell_n$.

The isomorphism classes of such \mathcal{T} -modules $V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s are in one-to-one correspondence with the set of pairs of $\ell \in \mathbb{N} \cup \{0\}$ and the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings that have the properties (i.1), (i.2), (i.3) above.

(ii) $\mathcal{M}_d^{s,t}(\mathcal{T})$ is nonempty if and only if the condition (16) holds. Suppose the condition (16) holds. Then $\mathcal{M}_d^{s,t}(\mathcal{T})$ consists of the isomorphism classes of \mathcal{T} -modules $V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s with the properties (i.1), (i.2), (i.3) above and

(ii.1) $-t^2 \notin S(\ell_i, a_i)$ for all i ($1 \leq i \leq n$).

The next theorem follows from Corollary 1.19 and [8, Proposition 7.15]. It solves Problem 1, which is to determine $\mathcal{Irr}(\mathcal{A})$, the set of isomorphism classes of finite-dimensional irreducible representations of the TD-algebra $\mathcal{A} = \mathcal{A}_q^{(1,0)}$ of the 2nd kind that have the property (C₁). For an \mathcal{L}' -module V , let ρ_V denote the representation of \mathcal{L}' afforded by the \mathcal{L}' -module V . Then $\rho_V \circ \varphi_s$ is the representation of \mathcal{T} afforded by the \mathcal{T} -module V via φ_s , and $\rho_V \circ \varphi_s \circ \iota_t$ is a representation of \mathcal{A} , where $\iota_t : \mathcal{A} \longrightarrow \mathcal{T}$ ($z, z^* \mapsto z_t, z_t^*$) is the injective algebra homomorphism from Proposition 1.1.

Theorem 1.20 *Assume $(\varepsilon, \varepsilon^*) = (1, 0)$. Let $\mathcal{A} = \mathcal{A}_q^{(1,0)}$ denote the TD-algebra of the 2nd kind. The following (i), (ii), (iii) hold.*

(i) For an \mathcal{L}' -module $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ and nonzero $s, t \in \mathbb{C}$, the representation $\rho_V \circ \varphi_s \circ \iota_t$ of \mathcal{A} is irreducible if and only if

(i.1) the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is in general position,

(i.2) none of $-s^{-2}, -t^2$ belongs to $S(\ell_i, a_i)$ for any i ($1 \leq i \leq n$),

(i.3) none of $\pm st$ equals q^i for any $i \in \mathbb{Z}$ ($-d+1 \leq i \leq d-1$).

(ii) For \mathcal{L}' -modules $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell') \otimes V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_{n'}, a'_{n'})$ and $(s, t), (s', t') \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$, set $\rho = \rho_V \circ \varphi_s \circ \iota_t$ and $\rho' = \rho_{V'} \circ \varphi_{s'} \circ \iota_{t'}$. Assume that the representations ρ, ρ' of \mathcal{A} are both irreducible. Then they are isomorphic as representations of \mathcal{A} if and only if $\ell = \ell'$, the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ coincide and $(s, t) \approx (s', t')$ in the sense of (12), i.e.,

$$(s', t') \in \{\pm(s, t), \pm(t^{-1}, s^{-1})\}.$$

(iii) Every nontrivial finite-dimensional irreducible representation of \mathcal{A} with the property (C_1) is isomorphic to $\rho_V \circ \varphi_s \circ \iota_t$ for some \mathcal{L}' -module $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ and $(s, t) \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$.

We do not give a proof of Theorem 1.20, since it can be proved by the same argument for the case of the q -Onsager algebra.

Finally we consider the case $(\varepsilon, \varepsilon^*) = (0, 0)$. By Proposition 1.13, φ_s gives an isomorphism between the augmented TD-algebra \mathcal{T} and the Borel subalgebra \mathcal{B} of \mathcal{L} . The TD-algebra \mathcal{A} is isomorphic to the positive part of the Borel subalgebra \mathcal{B} .

Theorem 1.21 (Case $(\varepsilon, \varepsilon^*) = (0, 0)$) Let $\mathcal{T} = \mathcal{T}_q^{(0,0)}$ denote the augmented TD-algebra of the 3rd kind. The following (i), (ii), (iii) hold.

(i) A tensor product $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules for \mathcal{L} is irreducible as a \mathcal{T} -module via φ_s if and only if the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is in general position. In this case, the \mathcal{T} -module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s has type s and diameter $d = \ell_1 + \cdots + \ell_n$ and the Drinfel'd polynomial $P_V(\lambda)$ of the \mathcal{T} -module V via φ_s is

$$P_V(\lambda) = \prod_{i=1}^n P_{V(\ell_i, a_i)}(\lambda),$$

where

$$P_{V(\ell_i, a_i)}(\lambda) = \prod_{c \in S(\ell_i, a_i)} (\lambda + c).$$

(ii) Set $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_{n'}, a'_{n'})$ and assume that V, V' are both irreducible as a \mathcal{T} -module via φ_s . Then V, V' are isomorphic as \mathcal{T} -modules via φ_s if and only if the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ of q -strings coincide.

(iii) Every nontrivial finite-dimensional irreducible \mathcal{T} -module of type s is isomorphic to some \mathcal{T} -module $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s .

Theorem 1.21 is well-known but a brief proof will be given in Section 7. The polynomial $\lambda^d P_V(\lambda^{-1})$ ($d = \ell_1 + \cdots + \ell_n$) for the case $(\varepsilon, \varepsilon^*) = (0, 0)$ is known as the original Drinfel'd polynomial:

$$\lambda^d P_V(\lambda^{-1}) = \prod_{i=1}^n \prod_{c \in S(\ell_i, a_i)} (1 + c\lambda).$$

Corollary 1.22 and Theorem 1.23 below, which are the main results of [7, Theorem 1.6, Theorem 1.7], follow immediately from Theorem 1.21 through Theorem 1.9' and Corollary 1.12, solving Problem 1 and Problem 2 in the case of $(\varepsilon, \varepsilon^*) = (0, 0)$.

Corollary 1.22 *Assume $(\varepsilon, \varepsilon^*) = (0, 0)$. Then $\mathcal{M}_d^s(\mathcal{T})$ and $\mathcal{M}_d^{s,t}(\mathcal{T})$ are determined as follows.*

(i) $\mathcal{M}_d^s(\mathcal{T})$ consists of the isomorphism classes of \mathcal{T} -modules $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s with the properties that

(i.1) the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is in general position,

(i.2) $d = \ell_1 + \cdots + \ell_n$.

The isomorphism classes of such \mathcal{T} -modules $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s are in one-to-one correspondence with the set of the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings that have the properties (i.1), (i.2) above.

(ii) $\mathcal{M}_d^{s,t}(\mathcal{T})$ is nonempty for any nonzero $s, t \in \mathbb{C}$. $\mathcal{M}_d^{s,t}(\mathcal{T})$ consists of the isomorphism classes of \mathcal{T} -modules $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s with the properties (i.1), (i.2) above and

(2.1) $-t^2 \notin S(\ell_i, a_i)$ for all i ($1 \leq i \leq n$).

Theorem 1.23 *Assume $(\varepsilon, \varepsilon^*) = (0, 0)$. Let $\mathcal{A} = \mathcal{A}_q^{(0,0)}$ denote the TD-algebra of the 3rd kind. The following (i), (ii), (iii) hold.*

(i) For an \mathcal{L} -module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ and nonzero $s, t \in \mathbb{C}$, the representation $\rho_V \circ \varphi_s \circ \iota_t$ of \mathcal{A} is irreducible if and only if

(i.1) the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is in general position,

(i.2) $-t^2 \notin S(\ell_i, a_i)$ for any i ($0 \leq i \leq n$).

(ii) For \mathcal{L} -modules $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_n, a'_n)$ and $(s, t), (s', t') \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$, set $\rho = \rho_V \circ \varphi_s \circ \iota_t$ and $\rho' = \rho_{V'} \circ \varphi_{s'} \circ \iota_{t'}$. Assume that the representations ρ, ρ' of \mathcal{A} are both irreducible. Then they are isomorphic as representations of \mathcal{A} if and only if the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ coincide and $(s, t) = \pm(s', t')$.

(iii) Every nontrivial finite-dimensional irreducible representation of \mathcal{A} with the property (C_1) is isomorphic to $\rho_V \circ \varphi_s \circ \iota_t$ for some \mathcal{L} -module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ and $(s, t) \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$.

Let $A, A^* \in \text{End}(V)$ be a TD-pair of Type I with eigenspaces $\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d$ respectively. Then we have the split decomposition (see Section 1.1):

$$V = \bigoplus_{i=0}^d U_i,$$

where

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d).$$

By [3, Corollary 5.7], it holds that

$$\dim U_i = \dim V_i = \dim V_i^* \quad (0 \leq i \leq d),$$

and

$$\dim U_i = \dim U_{d-i} \quad (0 \leq i \leq d).$$

Note that $\dim U_i$ is invariant under standardization of A, A^* . We want to find the generating function for $\dim U_i$:

$$g(\lambda) = \sum_{i=0}^d (\dim U_i) \lambda^i.$$

We may assume that A, A^* are standardized. Then by Theorem 1.17, Theorem 1.20, Theorem 1.23, the TD-pair A, A^* is afforded via $\varphi_s \circ \iota_t$ by an \mathcal{L} -module

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

for the cases $(\varepsilon, \varepsilon^*) = (1, 1), (0, 0)$ and by an \mathcal{L}' -module

$$V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

for the case $(\varepsilon, \varepsilon^*) = (1, 0)$. The split decomposition of V for A, A^* coincides with the eigenspace decomposition of the element k_0 of \mathcal{L} acting on V . Thus we have

Proposition 1.24 ([3, Conjecture 13.7])

$$\begin{aligned} g(\lambda) &= \prod_{i=1}^n (1 + \lambda + \lambda^2 + \cdots + \lambda^{\ell_i}) \quad \text{if } (\varepsilon, \varepsilon^*) = (1, 1), (0, 0), \\ g(\lambda) &= \prod_{i=0}^n (1 + \lambda + \lambda^2 + \cdots + \lambda^{\ell_i}) \quad \text{with } \ell_0 = \ell \quad \text{if } (\varepsilon, \varepsilon^*) = (1, 0). \end{aligned}$$

A TD-pair A, A^* is called a *Leonard pair* if $\dim U_i = 1$ for all i ($0 \leq i \leq d$). A standardized TD-pair A, A^* of Type I is a Leonard pair if and only if it is afforded by an evaluation module. In view of this fact, a standardized TD-pair A, A^* of Type I is regarded as a ‘tensor product of Leonard pairs’.

2 Linear bases for \mathcal{A} and \mathcal{T}

In this section, we give a linear basis for the TD-algebra \mathcal{A} that involves the generators z, z^* . We also give two linear bases for the augmented TD-algebra \mathcal{T} ; one involves the generators $x, y, k^{\pm 1}$ and the other involves the generators $z_t, z_t^*, k^{\pm 1}$ (see Section 1.2). Using these bases, we prove Proposition 1.1, Proposition 1.3 and Proposition 1.13.

For an integer $r \geq 0$, we denote by Λ_r the set of sequences $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_r)$ of integers such that $\lambda_0 \geq 0$, $\lambda_i \geq 1$ ($1 \leq i \leq r$), and define Λ to be the union of Λ_r ($r \geq 0$):

$$\begin{aligned}\Lambda_r &= \{\lambda = (\lambda_0, \lambda_1, \dots, \lambda_r) \in \mathbb{Z}^{r+1} \mid \lambda_0 \geq 0, \lambda_i \geq 1 \ (1 \leq i \leq r)\}, \\ \Lambda &= \bigcup_{r \in \mathbb{N} \cup \{0\}} \Lambda_r.\end{aligned}$$

Call $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_r) \in \Lambda$ *irreducible* if there exists an integer i ($0 \leq i \leq r$) such that

$$\lambda_0 < \lambda_1 < \dots < \lambda_i \geq \lambda_{i+1} \geq \dots \geq \lambda_r.$$

Note that each λ in $\Lambda_0 \cup \Lambda_1$ is irreducible. We denote the set of irreducible $\lambda \in \Lambda$ by Λ^{irr} :

$$\Lambda^{irr} = \{\lambda \in \Lambda \mid \lambda \text{ is irreducible}\}.$$

Let X, Y denote noncommuting indeterminates. For $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_r) \in \Lambda$, we define the word $\omega_\lambda(X, Y)$ by

$$\omega_\lambda(X, Y) = \begin{cases} X^{\lambda_0} Y^{\lambda_1} \dots X^{\lambda_r} & \text{if } r \text{ is even,} \\ X^{\lambda_0} Y^{\lambda_1} \dots Y^{\lambda_r} & \text{if } r \text{ is odd,} \end{cases}$$

where we interpret $X^{\lambda_0} = 1$ if $\lambda_0 = 0$. By the length of the word $\omega_\lambda(X, Y)$, we mean $\lambda_0 + \lambda_1 + \dots + \lambda_r$ and denote it by $|\lambda|$:

$$|\lambda| = \lambda_0 + \lambda_1 + \dots + \lambda_r.$$

Theorem 2.1 *The following set is a basis of the TD-algebra \mathcal{A} as a \mathbb{C} -vector space:*

$$\{\omega_\lambda(z, z^*) \mid \lambda \in \Lambda^{irr}\}.$$

Theorem 2.2 *Each of the following sets is a basis of the augmented TD-algebra \mathcal{T} as a \mathbb{C} -vector space:*

- (i) $\{k^n \omega_\lambda(x, y) \mid n \in \mathbb{Z}, \lambda \in \Lambda^{irr}\}.$
- (ii) $\{k^n \omega_\lambda(z_t, z_t^*) \mid n \in \mathbb{Z}, \lambda \in \Lambda^{irr}\}$, where t is a fixed nonzero scalar of \mathbb{C} and $z_t, z_t^*, k^{\pm 1}$ are the second generators of \mathcal{T} that are introduced in (3), (4) in Section 1.2.

We first prove the spanning property for the sets in Theorem 2.1, Theorem 2.2. Our strategy will be to reduce the essential part to [4, Theorem 2.29]. We start with a description of the \mathbb{C} -algebra generated by symbols $\xi, \eta, \kappa, \kappa^{-1}$ subject to the relations $(\text{TD})'_0$: $\kappa\kappa^{-1} = \kappa^{-1}\kappa = 1$, $\kappa\xi\kappa^{-1} = q^2\xi$, $\kappa\eta\kappa^{-1} = q^{-2}\eta$. Let Φ denote the free algebra over \mathbb{C} generated by symbols ξ, η . Let $\mathbb{C}[\kappa, \kappa^{-1}]$ denote the algebra over \mathbb{C} generated by symbols κ, κ^{-1} subject to the relations $\kappa\kappa^{-1} = \kappa^{-1}\kappa = 1$. Consider the \mathbb{C} -vector space $\mathbb{C}[\kappa, \kappa^{-1}] \otimes \Phi$, where $\otimes = \otimes_{\mathbb{C}}$. This space has the basis

$$\{\kappa^n \otimes \omega_\lambda(\xi, \eta) \mid n \in \mathbb{Z}, \lambda \in \Lambda\}.$$

Define the product of basis elements by

$$(\kappa^m \otimes \omega_\lambda(\xi, \eta))(\kappa^n \otimes \omega_\mu(\xi, \eta)) = \kappa^{m+n} \otimes \omega_\lambda(q^{-2n}\xi, q^{2n}\eta)\omega_\mu(\xi, \eta)$$

and extend it bilinearly to the product of elements of $\mathbb{C}[\kappa, \kappa^{-1}] \otimes \Phi$. Then $\mathbb{C}[\kappa, \kappa^{-1}] \otimes \Phi$ becomes an associative \mathbb{C} -algebra. The mapping $f \otimes u \mapsto fu$ ($f \in \mathbb{C}[\kappa, \kappa^{-1}]$, $u \in \Phi$) induces a \mathbb{C} -algebra isomorphism from $\mathbb{C}[\kappa, \kappa^{-1}] \otimes \Phi$ to the \mathbb{C} -algebra generated by $\xi, \eta, \kappa, \kappa^{-1}$ subject to the relations $(\text{TD})'_0$: $\kappa\kappa^{-1} = \kappa^{-1}\kappa = 1$, $\kappa\xi\kappa^{-1} = q^2\xi$, $\kappa\eta\kappa^{-1} = q^{-2}\eta$. We henceforth identify these two algebras via the isomorphism and denote this algebra by $\mathbb{C}[\kappa, \kappa^{-1}]\Phi$.

Define the elements $v_0, v_1 \in \Phi$ and $u_0, u_1 \in \mathbb{C}[\kappa, \kappa^{-1}]\Phi$ by

$$\begin{aligned} v_0 &= [\xi, \xi^2\eta - \beta\xi\eta\xi + \eta\xi^2], \\ v_1 &= [\xi\eta^2 - \beta\eta\xi\eta + \eta^2\xi, \eta], \\ u_0 &= \delta'(\varepsilon^*\xi^2k^2 - \varepsilon k^{-2}\xi^2), \\ u_1 &= \delta'(\varepsilon^*k^2\eta^2 - \varepsilon\eta^2k^{-2}), \end{aligned}$$

where $\beta = q^2 + q^{-2}$ and $\delta' = -(q - q^{-1})(q^2 - q^{-2})(q^3 - q^{-3})q^4$. Let \mathcal{I} denote the two-sided ideal of $\mathbb{C}[\kappa, \kappa^{-1}]\Phi$ generated by $v_0 - u_0, v_1 - u_1$:

$$\mathcal{I} = \mathbb{C}[\kappa, \kappa^{-1}]\Phi(v_0 - u_0)\Phi + \mathbb{C}[\kappa, \kappa^{-1}]\Phi(v_1 - u_1)\Phi.$$

Since the relations $(\text{TD})'$ for the augmented TD-algebra is $v_0 = u_0, v_1 = u_1$, the quotient algebra $\mathbb{C}[\kappa, \kappa^{-1}]\Phi/\mathcal{I}$ coincides with \mathcal{T} and we have the canonical algebra homomorphism

$$\pi_{\mathcal{T}} : \mathbb{C}[\kappa, \kappa^{-1}]\Phi \longrightarrow \mathcal{T} \quad (\xi, \eta, \kappa, \kappa^{-1} \mapsto x, y, k, k^{-1} \text{ respectively}).$$

Let \mathcal{J} denote the two-sided ideal of Φ generated by v_0, v_1 :

$$\mathcal{J} = \Phi v_0 \Phi + \Phi v_1 \Phi.$$

Write $\mathcal{A}_{\text{III}} = \Phi/\mathcal{J}$ for the quotient algebra (the TD-algebra of the 3rd kind), and let us use the bar notation for the canonical algebra homomorphism:

$$\pi_{\mathcal{A}_{\text{III}}} : \Phi \longrightarrow \mathcal{A}_{\text{III}} \quad (\xi, \eta \mapsto \bar{\xi}, \bar{\eta} \text{ respectively}).$$

By [4, Theorem 2.29], the set $\{\omega_\lambda(\bar{\xi}, \bar{\eta}) \mid \lambda \in \Lambda^{irr}\}$ is a basis for \mathcal{A}_{III} . Consequently

$$\Phi = \mathcal{W} + \mathcal{J} \quad (\text{direct sum}),$$

where \mathcal{W} is the subspace of Φ spanned by

$$\{\omega_\lambda(\xi, \eta) \mid \lambda \in \Lambda^{irr}\}.$$

For an integer $n \geq 0$, we mean by a word of length n in Φ a product $a_1 a_2 \cdots a_n$ in Φ such that $a_i \in \{\xi, \eta\}$ for $1 \leq i \leq n$. We interpret the word of length 0 as the identity in Φ . Let Φ_n denote the subspace of Φ spanned by the words of length n . For example, $\Phi_0 = \mathbb{C}1$. We have the direct sum $\Phi = \sum_{n \geq 0} \Phi_n$ and $\Phi_r \Phi_s = \Phi_{r+s}$ for all $r, s \geq 0$. For an integer $n \geq 0$, define $\mathcal{W}_n = \Phi_n \cap \mathcal{W}$ and $\mathcal{J}_n = \Phi_n \cap \mathcal{J}$. This yields the direct sum decompositions $\mathcal{W} = \sum_{n \geq 0} \mathcal{W}_n$ and $\mathcal{J} = \sum_{n \geq 0} \mathcal{J}_n$. By $\Phi = \mathcal{W} + \mathcal{J}$, we have

$$\Phi_n = \mathcal{W}_n + \mathcal{J}_n$$

for $n \geq 0$. Since $v_0, v_1 \in \Phi_4$,

$$\mathcal{J}_n = \sum \Phi_i v_0 \Phi_j + \sum \Phi_i v_1 \Phi_j,$$

where both sums are over the ordered pairs of nonnegative integers (i, j) such that $i + j = n - 4$. In particular, $\mathcal{J}_n = 0$ for $n \leq 3$. Since $v_0 = (v_0 - u_0) + u_0$, $v_1 = (v_1 - u_1) + u_1$ and $u_0, u_1 \in \mathbb{C}[\kappa, \kappa^{-1}] \Phi_2$, the above expression for \mathcal{J}_n together with the definition of \mathcal{I} implies

$$\mathcal{J}_n \subseteq \mathcal{I} + \mathbb{C}[\kappa, \kappa^{-1}] \Phi_{n-2} \quad (n \geq 4).$$

To prove that the set in Theorem 2.2 (i) spans \mathcal{T} , it suffices to show

$$\mathbb{C}[\kappa, \kappa^{-1}] \Phi = \mathbb{C}[\kappa, \kappa^{-1}] \mathcal{W} + \mathcal{I}.$$

To this end we show $\mathbb{C}[\kappa, \kappa^{-1}] \Phi_n \subseteq \mathbb{C}[\kappa, \kappa^{-1}] \mathcal{W} + \mathcal{I}$ for $n \geq 0$ and this will be done by induction on n . Let n be given. Recall $\Phi_n = \mathcal{W}_n + \mathcal{J}_n$. If $n \leq 3$, then $\mathcal{J}_n = 0$, and so $\Phi_n = \mathcal{W}_n \subseteq \mathcal{W}$ and certainly $\mathbb{C}[\kappa, \kappa^{-1}] \Phi_n \subseteq \mathbb{C}[\kappa, \kappa^{-1}] \mathcal{W} + \mathcal{I}$ as desired. If $n \geq 4$, we argue by $\mathcal{J}_n \subseteq \mathcal{I} + \mathbb{C}[\kappa, \kappa^{-1}] \Phi_{n-2}$

$$\begin{aligned} \mathbb{C}[\kappa, \kappa^{-1}] \Phi_n &= \mathbb{C}[\kappa, \kappa^{-1}] \mathcal{W}_n + \mathbb{C}[\kappa, \kappa^{-1}] \mathcal{J}_n \\ &\subseteq \mathbb{C}[\kappa, \kappa^{-1}] \mathcal{W} + \mathcal{I} + \mathbb{C}[\kappa, \kappa^{-1}] \Phi_{n-2} \end{aligned}$$

and this is contained in $\mathbb{C}[\kappa, \kappa^{-1}] \mathcal{W} + \mathcal{I}$ by induction on n . We have now proved that the set in Theorem 2.2 (i) spans \mathcal{T} .

To prove the spanning property for the sets in Theorem 2.1 and Theorem 2.2 (ii), let \mathcal{J}' denote the two-sided ideal of Φ generated by $v_0 - \varepsilon \delta[\xi, \eta]$ and $v_1 - \varepsilon^* \delta[\xi, \eta]$:

$$\mathcal{J}' = \Phi(v_0 - \varepsilon \delta[\xi, \eta])\Phi + \Phi(v_1 - \varepsilon^* \delta[\xi, \eta])\Phi,$$

where $\delta = -(q^2 - q^{-2})^2$. Since $\mathcal{J}_n = \sum \Phi_i v_0 \Phi_j + \sum \Phi_i v_1 \Phi_j$ over (i, j) with $i + j = n - 4$, we have

$$\mathcal{J}_n \subseteq \mathcal{J}' + \Phi_{n-2} \quad (n \geq 4),$$

noting that $[\xi, \eta] \in \Phi_2$. We claim that

$$\Phi = \mathcal{W} + \mathcal{J}'.$$

The inclusion \supseteq is from the construction. To get the inclusion \subseteq , we show $\Phi_n \subseteq \mathcal{W} + \mathcal{J}'$ for $n \geq 0$ and this will be done by induction on n . Let n be given. If $n \leq 3$, then $\mathcal{J}_n = 0$, $\Phi_n = \mathcal{W}_n + \mathcal{J}_n = \mathcal{W}_n \subseteq \mathcal{W}$ so $\Phi_n \subseteq \mathcal{W} + \mathcal{J}'$ as desired. If $n \geq 4$, we argue by $\mathcal{J}_n \subseteq \mathcal{J}' + \Phi_{n-2}$

$$\begin{aligned} \Phi_n &= \mathcal{W}_n + \mathcal{J}_n \\ &\subseteq \mathcal{W} + \mathcal{J}' + \Phi_{n-2} \end{aligned}$$

and this is contained in $\mathcal{W} + \mathcal{J}'$ by induction on n . We have now proved the claim. Write $\mathcal{A}' = \Phi/\mathcal{J}'$ for the quotient algebra, and let us use the prime notation for the canonical algebra homomorphism :

$$\pi_{\mathcal{A}'} : \Phi \longrightarrow \mathcal{A}' \quad (\xi, \eta \mapsto \xi', \eta' \text{ respectively}).$$

The above claim implies that \mathcal{A}' is spanned by

$$\{\omega_\lambda(\xi', \eta') \mid \lambda \in \Lambda^{irr}\}.$$

Since the defining relations (TD) for \mathcal{A} are $v_0 = \varepsilon\delta[\xi, \eta]$, $v_1 = \varepsilon^*\delta[\xi, \eta]$, \mathcal{A}' is isomorphic to the TD-algebra \mathcal{A} by the correspondence $\xi' \mapsto z$, $\eta' \mapsto z^*$. This proves that the set in Theorem 2.1 spans \mathcal{A} . For a fixed nonzero $t \in \mathbb{C}$, let $\langle z, z_t^* \rangle$ denote the subalgebra of \mathcal{T} generated by z_t, z_t^* . We have a surjective algebra homomorphism

$$\mathcal{A}' \longrightarrow \langle z_t, z_t^* \rangle \quad (\xi', \eta' \mapsto z_t, z_t^* \text{ respectively})$$

by the relations (TD) for z_t, z_t^* . So $\langle z_t, z_t^* \rangle$ is spanned by $\{\omega_\lambda(z_t, z_t^*) \mid \lambda \in \Lambda^{irr}\}$. By the relations (TD)₀ for \mathcal{T} , it holds that $\mathcal{T} = \sum_{n \in \mathbb{Z}} k^n \langle z_t, z_t^* \rangle$. Therefore the spanning property holds for the set in Theorem 2.2 (ii).

Next we prove the linear independency of the sets in Theorem 2.1 and Theorem 2.2. For a nonzero $s \in \mathbb{C}$, let φ_s be the algebra homomorphism from \mathcal{T} to the $U_q(sl_2)$ -loop algebra $\mathcal{L} = U_q(L(sl_2))$ as in Proposition 1.13: φ_s sends x, y, k to $x(s), y(s), k(s)$, respectively, where

$$\begin{aligned} x(s) &= \alpha(se_0^+ + \varepsilon s^{-1}e_1^-k_1) \quad \text{with } \alpha = -q^{-1}(q - q^{-1})^2, \\ y(s) &= \varepsilon^*se_0^-k_0 + s^{-1}e_1^+, \\ k(s) &= sk_0. \end{aligned}$$

Note that the existence of the algebra homomorphism φ_s has been established already, although the injectivity of φ_s is left to be proved. For a nonzero $t \in \mathbb{C}$, let ι_t be the algebra homomorphism from \mathcal{A} to \mathcal{T} as in Proposition 1.1: ι_t sends z, z^* to $z_t = x + tk + \varepsilon t^{-1}k^{-1}$, $z_t^* = y + \varepsilon^*t^{-1}k + tk^{-1}$, respectively. Note also that the existence of the algebra homomorphism ι_t has been established already, although the injectivity of ι_t is left to be proved. We set $z_t(s) = \varphi_s \circ \iota_t(z)$, $z_t^*(s) = \varphi_s \circ \iota_t(z^*)$:

$$\begin{aligned} z_t(s) &= x(s) + tk(s) + \varepsilon t^{-1}k(s)^{-1}, \\ z_t^*(s) &= y(s) + \varepsilon^*t^{-1}k(s) + tk(s)^{-1}. \end{aligned}$$

Lemma 2.3 For nonzero scalars $s, t \in \mathbb{C}$, each of the following sets is linearly independent in \mathcal{L} .

$$(i) \{k(s)^n \omega_\lambda(x(s), y(s)) \mid n \in \mathbb{Z}, \lambda \in \Lambda^{irr}\}.$$

$$(ii) \{k(s)^n \omega_\lambda(z_t(s), z_t^*(s)) \mid n \in \mathbb{Z}, \lambda \in \Lambda^{irr}\}.$$

The linear independency of the sets in Theorem 2.2 (resp. Theorem 2.1) immediately follows from Lemma 2.3 by applying the algebra homomorphism φ_s (resp. $\varphi_s \circ \iota_t$). We prove Lemma 2.3 by using the triangular decomposition of \mathcal{L} together with the basis of \mathcal{A}_{III} given in [4]. Let $\langle e_0^+, e_1^+ \rangle$ (resp. $\langle e_0^-, e_1^- \rangle$) be the subalgebra of \mathcal{L} generated by e_0^+, e_1^+ (resp. e_0^-, e_1^-). Then by [4, Theorem 2.29], $\langle e_0^+, e_1^+ \rangle$ (resp. $\langle e_0^-, e_1^- \rangle$) is isomorphic to \mathcal{A}_{III} and has the set B^+ (resp. B^-) as a linear basis, where

$$\begin{aligned} B^+ &= \{\omega_\lambda(e_0^+, e_1^+) \mid \lambda \in \Lambda^{irr}\}, \\ B^- &= \{\omega_\lambda(e_0^-, e_1^-) \mid \lambda \in \Lambda^{irr}\}. \end{aligned}$$

By the triangular decomposition of \mathcal{L} , the set

$$B = \{\omega^- k_0^n \omega^+ \mid n \in \mathbb{Z}, \omega^- \in B^-, \omega^+ \in B^+\}$$

is a linear basis of \mathcal{L} , and so every element of \mathcal{L} is uniquely expressed as a finite sum of

$$c_{\mu, n, \lambda} \omega_\mu(e_0^-, e_1^-) k_0^n \omega_\lambda(e_0^+, e_1^+)$$

with $c_{\mu, n, \lambda} \in \mathbb{C}$, $n \in \mathbb{Z}$, $\mu, \lambda \in \Lambda^{irr}$. The expression for the element $k(s)^n \omega_\lambda(x(s), y(s))$ in question of Lemma 2.3 (i) is, by the defining relations of \mathcal{L} ,

$$s^n k_0^n \omega_\lambda(\alpha s e_0^+, s^{-1} e_1^+)$$

plus some other terms $c_{\mu', n'} \omega_{\mu'}(e_0^-, e_1^-) k_0^{n'} \omega_{\lambda'}(e_0^+, e_1^+)$ with $|\lambda'| < |\lambda|$. The highest term $s^n k_0^n \omega_\lambda(\alpha s e_0^+, s^{-1} e_1^+)$ is the product of the nonzero scalar $s^n \omega_\lambda(\alpha s, s^{-1}) \in \mathbb{C}$, k_0^n and the element $\omega_\lambda(e_0^+, e_1^+) \in B^+$. Therefore any linear dependency relation among the elements in Lemma 2.3 (i) is the trivial one by induction on the maximal length $|\lambda|$ of λ that appears in the relation. Similarly the set in (ii) is shown to be linearly independent. This completes the proof of Lemma 2.3.

Proof of Proposition 1.1 and Proposition 1.13: the injectivity of ι_t and φ_s . The algebra homomorphism $\varphi_s \circ \iota_t$ is injective by Theorem 2.1 and Lemma 2.3 (ii) and so ι_t is injective. The injectivity of φ_s follows from Theorem 2.2 (i) and Lemma 2.3 (i). \square

Proof of Proposition 1.3. Let V be a finite-dimensional irreducible \mathcal{T} -module of type s , diameter d and $V = \bigoplus_{i=0}^d U_i$ the weight space decomposition from Lemma 1.2. Since $z_t = x + tk + \varepsilon t^{-1} k^{-1}$, $k|_{U_i} = sq^{2i-d}$, we have $z_t = x + \theta_i$ on U_i , where $\theta_i = stq^{2i-d} + \varepsilon s^{-1} t^{-1} q^{d-2i}$. Since $xU_i \subseteq U_{i+1}$, we have $(z_t - \theta_0)(z_t - \theta_1) \cdots (z_t - \theta_d) = 0$ on V . If $\theta_0, \dots, \theta_d$ are mutually distinct, then z_t is diagonalizable on V and it holds that

$$V_i + V_{i+1} + \cdots + V_d = U_i + U_{i+1} + \cdots + U_d \quad (0 \leq i \leq d),$$

where V_i is the eigenspace of z_t on V that belongs to the eigenvalue θ_i .

Conversely, suppose z_t is diagonalizable on V . Let $\theta_{i_0}, \theta_{i_1}, \dots, \theta_{i_r}$ denote the distinct members among θ_i ($0 \leq i \leq d$). Then $(z_t - \theta_{i_r}) \cdots (z_t - \theta_{i_1})(z_t - \theta_{i_0})$ vanishes on V , in particular on U_0 . We claim

$$(z_t - \theta_{i_j}) \cdots (z_t - \theta_{i_1})(z_t - \theta_{i_0}) = f_j(x) \quad \text{on } U_0$$

for some monic polynomial f_j of degree $j + 1$ ($0 \leq j \leq r$). The claim holds for $j = 0$, since $z_t - \theta_{i_0} = x + \theta_0 - \theta_{i_0}$ on U_0 . If the claim holds for j , then there exist scalars c_0, c_1, \dots, c_{j+1} with $c_{j+1} = 1$ such that

$$(z_t - \theta_{i_j}) \cdots (z_t - \theta_{i_1})(z_t - \theta_{i_0})u = \sum_{i=0}^{j+1} c_i x^i u \quad (u \in U_0).$$

Since the right-hand side has the i -th term $c_i x^i u \in U_i$ and $z_t - \theta_{i_{j+1}} = x + \theta_i - \theta_{i_{j+1}}$ on U_i , the claim holds for $j + 1$. Thus the claim is proved by induction on j . Since $(z_t - \theta_{i_r}) \cdots (z_t - \theta_{i_1})(z_t - \theta_{i_0})$ vanishes on U_0 , the monic polynomial f_r of degree $r + 1$ satisfies $f_r(x)U_0 = 0$. This implies $x^{r+1}U_0 = 0$, since $x^i U_0 \subseteq U_i$ and V is the direct sum of U_i 's. On the other hand, we have $V = \mathcal{T}U_0$ by the irreducibility of the \mathcal{T} -module V , so V is spanned by $\omega_\lambda(x, y)U_0$ ($\lambda \in \Lambda^{irr}$) due to Theorem 2.2. For $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_n) \in \Lambda^{irr}$, there exists some i ($0 \leq i \leq n$) such that $\lambda_0 < \lambda_1 < \dots < \lambda_i \geq \lambda_{i+1} \geq \dots \geq \lambda_n$. If $\omega_\lambda(x, y)U_0 \neq 0$ for such λ , then i, n are even and it holds that $\lambda_{i+1} = \lambda_{i+2} \geq \dots \geq \lambda_{n-1} = \lambda_n$, since $xU_j \subseteq U_{j+1}$, $yU_j \subseteq U_{j-1}$ with $U_{-1} = 0$. Moreover we have $\lambda_i \leq r$, otherwise $\omega_\lambda(x, y)U_0 = 0$ by the vanishing property $x^{r+1}U_0 = 0$ we just proved. Therefore if $\omega_\lambda(x, y)U_0 \neq 0$, then $\omega_\lambda(x, y)U_0 \subseteq U_j$, where $j = \lambda_i - \lambda_{i-1} + \dots + \lambda_2 - \lambda_1 + \lambda_0 \leq \lambda_i \leq r$. Thus $V = \mathcal{T}U_0 \subseteq U_0 + U_1 + \dots + U_r$. This implies $r = d$, i.e., $\theta_0, \dots, \theta_d$ are mutually distinct. We have now prove the first half (i) of Proposition 1.3. The second half (ii) is similarly proved, using $V = \mathcal{T}U_d$. \square

3 The subspace of height 0 in \mathcal{T}

Let \mathcal{T} be the augmented TD-algebra. \mathcal{T} is the algebra generated by x, y, k^\pm subject to the relations $(\text{TD})'_0, (\text{TD})'$ in Section 1.2. We introduce the notion of height for a word in x, y, k^\pm and discuss the structure of the subspace of \mathcal{T} spanned by the words of height 0. The main result of this section is Theorem 3.1. As applications of Theorem 3.1, we prove Theorem 1.8 and the injectivity of σ in Theorem 1.9. We keep the notations in Section 2.

Consider the free algebra over \mathbb{C} generated by $\xi, \eta, \kappa, \kappa^{-1}$. Let τ_0 denote the automorphism of this free algebra that sends $\xi, \eta, \kappa, \kappa^{-1}$ to $\eta, \xi, \kappa, \kappa^{-1}$ respectively, and let τ_1 denote the anti-automorphism that reverses the word order. Then τ_0, τ_1 commute and the product $\tau = \tau_0\tau_1 = \tau_1\tau_0$ is an antiautomorphism that sends a word $\zeta_1\zeta_2 \cdots \zeta_n$ to $\zeta'_n \cdots \zeta'_2\zeta'_1$ ($\zeta_i \in \{\xi, \eta, \kappa, \kappa^{-1}\}$), where $\zeta'_i = \eta, \xi, \kappa, \kappa^{-1}$ for $\zeta_i = \xi, \eta, \kappa, \kappa^{-1}$ respectively. Note that $\tau_0^2 = \tau_1^2 = \tau^2 = id$, the identity map. Keeping the notations in Section 2, let Φ denote the free algebra generated by ξ, η , and $\mathbb{C}[\kappa, \kappa^{-1}]\Phi$ the algebra generated by $\xi, \eta, \kappa, \kappa^{-1}$ subject

to the relations $(\text{TD})'_0 : \kappa\kappa^{-1} = \kappa^{-1}\kappa = 1, \kappa\xi\kappa^{-1} = q^2\xi, \kappa\eta\kappa^{-1} = q^{-2}\eta$. Since $(\text{TD})'_0$ is invariant under τ as a set of relations, the map τ induces an anti-automorphism of the algebra $\mathbb{C}[\kappa, \kappa^{-1}]\Phi$. Recall the elements $v_0, v_1 \in \Phi$ and $u_0, u_1 \in \mathbb{C}[k, k^{-1}]\Phi$ introduced in Section 2:

$$\begin{aligned} v_0 &= [\xi, \xi^2\eta - \beta\xi\eta\xi + \eta\xi^2], \\ v_1 &= [\xi\eta^2 - \beta\eta\xi\eta + \eta^2\xi, \eta], \\ u_0 &= \delta'(\varepsilon^*\xi^2\kappa^2 - \varepsilon\kappa^{-2}\xi^2), \\ u_1 &= \delta'(\varepsilon^*\kappa^2\eta^2 - \varepsilon\eta^2\kappa^{-2}), \end{aligned}$$

where $\beta = q^2 + q^{-2}$, $\delta' = -(q - q^{-1})(q^{-2} - q^2)(q^{-3} - q^3)q^4$. The augmented TD-algebra \mathcal{T} is defined by $(\text{TD})' : v_0 = u_0, v_1 = u_1$ together with $(\text{TD})'_0$. Since $v_0^\tau = v_1, u_0^\tau = u_1$, $(\text{TD})'$ is invariant under τ and the map τ induces an anti-automorphism of \mathcal{T} . Also τ induces an anti-automorphism of $\mathcal{A}_{\text{III}} = \Phi/\mathcal{J}$, where \mathcal{J} is the two-sided ideal of Φ generated by v_0, v_1 . We use the same notation τ for these anti-automorphisms of $\mathbb{C}[\kappa, \kappa^{-1}]\Phi$, \mathcal{T} , \mathcal{A}_{III} .

Let W denote the free semi-group generated by ξ, η . As a set, W is the collection of all words in Φ . Let h

$$h : W \longrightarrow \mathbb{Z}$$

denote the semi-group homomorphism from W to the additive group \mathbb{Z} defined by $h(\xi) = 1, h(\eta) = -1$. For a word $w \in W$, the value $h(w)$ is called the *height* of w . Thus a word of height 0 is a word in which ξ, η appear the same number of times. Denote by $\Phi^{(i)}$ the subspace of Φ linearly spanned by the words of height i :

$$\Phi^{(i)} = \text{Span}\{w \in W \mid h(w) = i\}.$$

Then Φ is the direct sum of the vector spaces $\Phi^{(i)}$ ($i \in \mathbb{Z}$):

$$\Phi = \bigoplus_{i \in \mathbb{Z}} \Phi^{(i)}. \quad (17)$$

The above decomposition is an algebra grading, i.e., $\Phi^{(i)}\Phi^{(j)} \subseteq \Phi^{(i+j)}$. Note that $\Phi^{(0)}$ is a subalgebra of Φ . The antiautomorphism τ changes the sign of the height of a word and so sends $\Phi^{(i)}$ to $\Phi^{(-i)}$. In particular, τ induces an antiautomorphism of the subalgebra $\Phi^{(0)}$. Let Φ^{sym} denote the subspace of $\Phi^{(0)}$ consisting of the fixed points of τ :

$$\Phi^{\text{sym}} = \{v \in \Phi^{(0)} \mid v^\tau = v\}.$$

A word $w \in W$ is called *nil* if w can be written as $w = w_1w_2$ with $w_1, w_2 \in W$ and $h(w_2) < 0$. Let Φ^{nil} denote the subspace of $\Phi^{(0)}$ linearly spanned by the words of height 0 that are nil:

$$\Phi^{\text{nil}} = \text{Span}\{w \in W \mid h(w) = 0, w \text{ is nil}\}.$$

Then Φ^{nil} is a two-sided ideal of $\Phi^{(0)}$ and invariant under the antiautomorphism τ . Recall Φ_n is the subspace of Φ spanned by the words of length n in ξ, η . Set $\Phi_n^{\text{sym}} = \Phi_n \cap \Phi^{\text{sym}}$ and $\Phi_n^{\text{nil}} = \Phi_n \cap \Phi^{\text{nil}}$. Then we have the direct sum decompositions as vector spaces:

$$\Phi^{\text{sym}} = \bigoplus_{n \geq 0} \Phi_n^{\text{sym}}, \quad (18)$$

$$\Phi^{\text{nil}} = \bigoplus_{n \geq 0} \Phi_n^{\text{nil}}. \quad (19)$$

The algebra $\mathbb{C}[\kappa, \kappa^{-1}]\Phi$ becomes a graded algebra

$$\mathbb{C}[\kappa, \kappa^{-1}]\Phi = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\kappa, \kappa^{-1}]\Phi^{(i)}.$$

Recall $\mathcal{T} = \mathbb{C}[\kappa, \kappa^{-1}]\Phi/\mathcal{I}$, where \mathcal{I} is the two-sided ideal of $\mathbb{C}[\kappa, \kappa^{-1}]\Phi$ generated by $v_0 - u_0$, $v_1 - u_1$. Note that $v_0 - u_0$, $v_1 - u_1$ belong to $\mathbb{C}[\kappa, \kappa^{-1}]\Phi^{(2)}$, $\mathbb{C}[\kappa, \kappa^{-1}]\Phi^{(-2)}$ respectively. Set

$$\mathcal{I}^{(i)} = \mathcal{I} \cap \mathbb{C}[\kappa, \kappa^{-1}]\Phi^{(i)}.$$

Then we have

$$\mathcal{I} = \bigoplus_{i \in \mathbb{Z}} \mathcal{I}^{(i)}. \quad (20)$$

For $\mathcal{T} = \mathbb{C}[\kappa, \kappa^{-1}]\Phi/\mathcal{I}$, consider the canonical homomorphism

$$\pi = \pi_{\mathcal{T}} : \mathbb{C}[\kappa, \kappa^{-1}]\Phi \longrightarrow \mathcal{T} \quad (\xi, \eta, \kappa, \kappa^{-1} \mapsto x, y, k, k^{-1} \text{ respectively}). \quad (21)$$

Set $\Psi = \pi(\Phi)$, $\Psi^{(i)} = \pi(\Phi^{(i)})$. Then by (20), the algebra \mathcal{T} inherits the algebra grading of $\mathbb{C}[\kappa, \kappa^{-1}]\Phi = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\kappa, \kappa^{-1}]\Phi^{(i)}$ via π :

$$\mathcal{T} = \mathbb{C}[k, k^{-1}]\Psi = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[k, k^{-1}]\Psi^{(i)}.$$

This enables us to define the *height* function for \mathcal{T} : a nonzero element of \mathcal{T} is said to have height i if it belongs to $\mathbb{C}[k, k^{-1}]\Psi^{(i)}$.

Note that $\Psi = \pi(\Phi)$ is the subalgebra of \mathcal{T} generated by x, y . Ψ has the grading

$$\Psi = \bigoplus_{i \in \mathbb{Z}} \Psi^{(i)}. \quad (22)$$

$\Psi^{(i)}$ is the subspace of Ψ spanned by the words in x, y of height i . $\Psi^{(0)}$ is a subalgebra of Ψ . The antiautomorphism τ of \mathcal{T} sends $\Psi^{(i)}$ to $\Psi^{(-i)}$. In particular, τ induces an antiautomorphism of the subalgebra $\Psi^{(0)}$. Set

$$\Psi^{sym} = \pi(\Phi^{sym}).$$

Then $\Psi^{sym} \subseteq \Psi^{(0)}$ and every element of Ψ^{sym} is fixed by τ . Let Ψ^{nil} denote the image of Φ^{nil} under π :

$$\Psi^{nil} = \pi(\Phi^{nil}).$$

Then Ψ^{nil} is a two-sided ideal of $\Psi^{(0)}$ and invariant under τ . Note that k, k^{-1} commute with every element of $\Psi^{(0)}$. So $\mathbb{C}[k, k^{-1}]\Psi^{(0)}$ is a subalgebra of \mathcal{T} and $\mathbb{C}[k, k^{-1}]\Psi^{sym}$ (resp. $\mathbb{C}[k, k^{-1}]\Psi^{nil}$) is a subspace (resp. two-sided ideal) of $\mathbb{C}[k, k^{-1}]\Psi^{(0)}$.

Theorem 3.1 *The following (i), (ii) hold.*

$$(i) \quad \mathbb{C}[k, k^{-1}]\Psi^{(0)} = \mathbb{C}[k, k^{-1}]\Psi^{sym} + \mathbb{C}[k, k^{-1}]\Psi^{nil}.$$

(ii) The quotient algebra $\mathbb{C}[k, k^{-1}]\Psi^{(0)}/\mathbb{C}[k, k^{-1}]\Psi^{nil}$ is commutative and generated by k, k^{-1} and $y^i x^i$ ($i = 0, 1, 2, \dots$) mod $\mathbb{C}[k, k^{-1}]\Psi^{nil}$.

Proof. Our strategy will be to reduce the essential part to [4, Theorem 2.20]. Recall the canonical homomorphism in Section 2

$$\pi_{\mathcal{A}_{III}} : \Phi \longrightarrow \mathcal{A}_{III} = \Phi/\mathcal{J} \quad (\xi, \eta \mapsto \bar{\xi}, \bar{\eta} \text{ respectively}),$$

where \mathcal{J} is the two-sided ideal of Φ generated by v_0, v_1 . Apply $\pi_{\mathcal{A}_{III}}$ to $\Phi^{(0)}$, Φ^{sym} , Φ^{nil} and denote the images by $\mathcal{A}^{(0)}$, \mathcal{A}^{sym} , \mathcal{A}^{nil} respectively. Then \mathcal{A}^{nil} is a two-sided ideal of $\mathcal{A}^{(0)}$ and the quotient $\mathcal{A}^{(0)}/\mathcal{A}^{nil}$ is a commutative algebra generated by $\bar{\eta}^i \bar{\xi}^i$ ($i = 0, 1, 2, \dots$) mod \mathcal{A}^{nil} (see [4, Lemma 3.1]). So each element of $\mathcal{A}^{(0)}/\mathcal{A}^{nil}$ is a linear combination of $(\bar{\eta}^{i_1} \bar{\xi}^{i_1})(\bar{\eta}^{i_2} \bar{\xi}^{i_2}) \dots (\bar{\eta}^{i_n} \bar{\xi}^{i_n})$ mod \mathcal{A}^{nil} . Apply the antiautomorphism τ of \mathcal{A}_{III} to $\bar{w} = (\bar{\eta}^{i_1} \bar{\xi}^{i_1})(\bar{\eta}^{i_2} \bar{\xi}^{i_2}) \dots (\bar{\eta}^{i_n} \bar{\xi}^{i_n})$. Then by the commutativity of $\mathcal{A}^{(0)}/\mathcal{A}^{nil}$, we have

$$\begin{aligned} \bar{w} - \bar{w}^\tau &= (\bar{\eta}^{i_1} \bar{\xi}^{i_1})(\bar{\eta}^{i_2} \bar{\xi}^{i_2}) \dots (\bar{\eta}^{i_n} \bar{\xi}^{i_n}) - (\bar{\eta}^{i_n} \bar{\xi}^{i_n}) \dots (\bar{\eta}^{i_2} \bar{\xi}^{i_2})(\bar{\eta}^{i_1} \bar{\xi}^{i_1}) \\ &\equiv 0 \text{ mod } \mathcal{A}^{nil}. \end{aligned}$$

Thus

$$\bar{w} = \frac{1}{2}(\bar{w} + \bar{w}^\tau) + \frac{1}{2}(\bar{w} - \bar{w}^\tau) \in \mathcal{A}^{sym} + \mathcal{A}^{nil},$$

and hence

$$\mathcal{A}^{(0)} = \mathcal{A}^{sym} + \mathcal{A}^{nil}.$$

This means that for a word w in ξ, η of length n , height 0, there exist elements $v \in \Phi^{sym}$, $v' \in \Phi^{nil}$ such that $w - v - v' \in \mathcal{J}$. Write v (resp. v') in the form of the direct sum (18) (resp. (19)): $v = \sum_i v_i$, $v' = \sum_i v'_i$ with $v_i \in \Phi_i^{sym}$, $v'_i \in \Phi_i^{nil}$. Observe $\mathcal{J} = \bigoplus_i \mathcal{J}_i$, where $\mathcal{J}_i = \mathcal{J} \cap \Phi_i$. Since $w \in \Phi_n$, we have $w - v_n - v'_n \in \mathcal{J}_n$. Thus from the beginning, we may assume $v = v_n$, $v' = v'_n$, i.e., for a word w in ξ, η of length n , height 0, there exist elements $v \in \Phi_n^{sym}$, $v' \in \Phi_n^{nil}$ such that

$$w - v - v' \in \mathcal{J}_n = \mathcal{J} \cap \Phi_n. \quad (23)$$

First we prove Theorem 3.1 (i). Take any word \hat{w} in x, y of height 0, length n . Choose a word w in ξ, η of height 0, length n such that $\hat{w} = \pi(w)$, where π is the canonical homomorphism from (21). Then by (23) there exist elements $v \in \Phi_n^{sym}$, $v' \in \Phi_n^{nil}$ such that $w - v - v' \in \mathcal{J}_n$. Observe $\mathcal{J}_n = \sum \Phi_i v_0 \Phi_j + \sum \Phi_i v_1 \Phi_j$, where the summation is over i, j with $i + j = n - 4$, since v_0, v_1 have length 4. Write the element $w - v - v' \in \mathcal{J}_n$ as a linear combination of $w_i v_0 w_j, w'_i v_1 w'_j$ for finitely many words w_i, w_j, w'_i, w'_j in ξ, η such that $\ell(w_i) + \ell(w_j) = \ell(w'_i) + \ell(w'_j) = n - 4$, where the function ℓ stands for the length of a word. Recall that Φ is a graded algebra according to the height as in (17). Since the element $w - v - v'$ has height 0 and v_0, v_1 have height 2, -2 respectively, we may assume that $h(w_i) + h(w_j) = -2$, $h(w'_i) + h(w'_j) = 2$, where the function h stands for the height of a word. Apply the canonical homomorphism π from (21) to $w - v - v'$. Then $\pi(v) \in \Psi^{sym}$, $\pi(v') \in \Psi^{nil}$. Since $\pi(v_0) = \pi(u_0)$, $\pi(v_1) = \pi(u_1)$, the terms $w_i v_0 w_j, w'_i v_1 w'_j$ in the linear combination for $w - v - v'$ are mapped to

$$\begin{aligned} \pi(w_i)\pi(u_0)\pi(w_j) &\in \mathbb{C}[k, k^{-1}](\Psi^{(0)} \cap \Psi_{n-2}), \\ \pi(w'_i)\pi(u_1)\pi(w'_j) &\in \mathbb{C}[k, k^{-1}](\Psi^{(0)} \cap \Psi_{n-2}), \end{aligned}$$

where $\Psi_m = \pi(\Phi_m)$. Thus $\pi(w) - \pi(v) - \pi(v')$ belongs to $\mathbb{C}[k, k^{-1}](\Psi^{(0)} \cap \Psi_{n-2})$, while $\pi(v) + \pi(v')$ belongs to $\Psi^{sym} + \Psi^{nil}$. The proof of part (i) is completed by induction on n .

Next we prove Theorem 3.1 (ii). By Theorem 2.2, $\mathbb{C}[k, k^{-1}]\Psi^{(0)}/\mathbb{C}[k, k^{-1}]\Psi^{nil}$ is linearly spanned by $k^n w_\lambda(x, y) \bmod \mathbb{C}[k, k^{-1}]\Psi^{nil}$, where n runs through \mathbb{Z} and λ runs through irreducible sequences such that the word $w_\lambda(x, y)$ has height 0. Since $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_r)$ is irreducible and $w_\lambda(x, y)$ has height 0, we may assume that r is even and $\lambda_0 = 0, \lambda_1 = \lambda_2 \geq \lambda_3 = \lambda_4 \geq \dots \geq \lambda_{r-1} = \lambda_r$, otherwise $w_\lambda(x, y) \in \Psi^{nil}$. Therefore $\mathbb{C}[k, k^{-1}]\Psi^{(0)}/\mathbb{C}[k, k^{-1}]\Psi^{nil}$ is generated by $k^{\pm 1}$ and $y^i x^i$ ($i = 0, 1, 2, \dots$) $\bmod \mathbb{C}[k, k^{-1}]\Psi^{nil}$. Note that $k^{\pm 1}$ commutes with $y^i x^i$. We want to show $y^i x^i, y^j x^j$ commute $\bmod \mathbb{C}[k, k^{-1}]\Psi^{nil}$. Set $w = (y^i x^i)(y^j x^j)$. By part (i) we just proved, there exist $f, g \in \mathbb{C}[k, k^{-1}]$, $u \in \Psi^{sym}$, $v \in \Psi^{nil}$ such that $w = fu + gv$. Then $w^\tau = fu + gv^\tau$, since $u^\tau = u$ and k, k^{-1} commute with every word of height 0 in x, y . Note that Ψ^{nil} is invariant under τ , so $w - w^\tau = g(v - v^\tau) \in \mathbb{C}[k, k^{-1}]\Psi^{nil}$. Since $w - w^\tau = (y^i x^i)(y^j x^j) - (y^j x^j)(y^i x^i)$, this means $y^i x^i, y^j x^j$ commute $\bmod \mathbb{C}[k, k^{-1}]\Psi^{nil}$ and the proof of part (ii) is completed. \square

Proof of Theorem 1.8. Let V be a finite-dimensional irreducible module of the augmented TD-algebra \mathcal{T} . Let $V = \bigoplus_{i=0}^d U_i$ denote the weight-space decomposition of the \mathcal{T} -module V . We want to show $\dim U_i \leq \binom{d}{i}$ ($0 \leq i \leq d$).

Recall the algebra grading $\mathcal{T} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[k, k^{-1}]\Psi^{(i)}$, where $\Psi^{(i)}$ is the linear span of the words of height i in x, y . Also recall $xU_i \subseteq U_{i+1}$, $yU_i \subseteq U_{i-1}$. The subalgebra $\mathbb{C}[k, k^{-1}]\Psi^{(0)}$ acts on U_0 and $\mathbb{C}[k, k^{-1}]\Psi^{nil}$ belongs to the kernel of the action. By Theorem 3.1 (ii), there exists a common eigenvector $v \in U_0$ of $y^i x^i$ ($0 \leq i \leq d$). Since $y^j x^j$ vanishes on U_0 for $j \geq d+1$, each element of $\mathbb{C}[k, k^{-1}]\Psi^{(0)}$ fixes the 1-dimensional subspace $\mathbb{C}v$ by Theorem 3.1 (ii). Since V is irreducible and $\mathcal{T} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[k, k^{-1}]\Psi^{(i)}$, we have $V = \mathcal{T}v = \sum_{i=0}^d \Psi^{(i)}v$. Then $U_i = \Psi^{(i)}v$, since $\Psi^{(i)}v \subseteq U_i$ and the sum $V = \sum_{i=0}^d U_i$ is direct. In particular, $U_0 = \Psi^{(0)}v = \mathbb{C}v$. By Theorem 2.2,

$$U_i = \Psi^{(i)}v = \sum_{\lambda \in \Lambda^{(i)}} \mathbb{C} \omega_\lambda(x, y) v, \quad (24)$$

where $\Lambda^{(i)}$ denotes the set of $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_r) \in \Lambda^{irr}$ such that r is even and

$$\sum_{j=0}^r (-1)^j \lambda_j = i, \quad \lambda_0 < \lambda_1 < \dots < \lambda_r \leq d.$$

Since $\Lambda^{(i)}$ contains exactly $\binom{d}{i}$ members, the proof of Theorem 1.8 is completed. \square

Proof of Theorem 1.9: the injectivity of σ . Let V be a finite-dimensional irreducible module of the augmented TD-algebra \mathcal{T} . Let $V = \bigoplus_{i=0}^d U_i$ denote the weight-space decomposition of the \mathcal{T} -module V . Recall $kv = sq^{2i-d}v$ for $v \in U_i$, $xU_i \subseteq U_{i+1}$, $yU_i \subseteq U_{i-1}$. By Theorem 1.8 we just proved, $\dim U_0 = 1$. Let $\sigma_i = \sigma_i(V)$ denote the eigenvalue of $y^i x^i$ on the heighest weight space U_0 . Apparently $\sigma_0 = 1, \sigma_i = 0$ for $i \geq d+1$.

We want to show $\sigma_d \neq 0$. By (24) in the proof of Theorem 1.8, it holds that $U_d = \sum_{\lambda \in \Lambda^{(d)}} \omega_\lambda(x, y) U_0$. Since $\Lambda^{(d)} = \{\lambda = (\lambda_0) \mid \lambda_0 = d\}$, we have $U_d = x^d U_0$. In the proof of

Theorem 1.8, the formula (24) follows from $V = \bigoplus_{i=0}^d \Psi^{(i)} U_0$. Apply the same argument starting with $V = \bigoplus_{i=0}^d \Psi^{(-i)} U_d$. Then we end up with $U_0 = \sum_{\lambda \in \Lambda^{(d)}} \omega_\lambda(y, x) U_d$. Thus we have $U_0 = y^d U_d$. So $U_0 = y^d U_d = y^d x^d U_0$ and the eigenvalue σ_d of $y^d x^d$ on U_0 is nonzero. Thus the diameter d of the \mathcal{T} -module V is determined by the property $\sigma_d \neq 0, \sigma_i = 0$ ($i \geq d+1$) of the sequence $\{\sigma_i\}_{i=0}^\infty$.

Next we want to show that the isomorphism class of the \mathcal{T} -module V is determined by the type s and the sequence $\{\sigma_i\}_{i=0}^d$. Let \mathcal{N} denote the set of elements of \mathcal{T} that vanish on U_0 :

$$\mathcal{N} = \{\nu \in \mathcal{T} \mid \nu U_0 = 0\}.$$

Then \mathcal{N} is a maximal left ideal of \mathcal{T} and V is isomorphic to \mathcal{T}/\mathcal{N} as \mathcal{T} -modules. Hence it is enough to show that \mathcal{N} is determined by s and $\{\sigma_i\}_{i=0}^d$. With respect to the algebra grading $\mathcal{T} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[k, k^{-1}] \Psi^{(i)}$, write $\nu \in \mathcal{N}$ as $\nu = \sum \nu_i$ ($\nu_i \in \mathbb{C}[k, k^{-1}] \Psi^{(i)}$). Then $\nu_i U_0 \subseteq U_i$. Since $V = \bigoplus_{i=0}^d U_i$ and $\nu U_0 = 0$, we have $\nu_i U_0 = 0$, i.e., $\nu_i \in \mathcal{N}$. Therefore

$$\mathcal{N} = \bigoplus_{i \in \mathbb{Z}} \mathcal{N}^{(i)},$$

where $\mathcal{N}^{(i)} = \mathcal{N} \cap \mathbb{C}[k, k^{-1}] \Psi^{(i)}$. Note that $\mathcal{N}^{(i)} = \mathbb{C}[\kappa, \kappa^{-1}] \Psi^{(i)}$ for $i < 0$. Thus it is enough to show that $\mathcal{N}^{(i)}$ is determined by s and $\{\sigma_j\}_{j=0}^d$ for $i = 0, 1, 2, \dots$.

For $i = 0$, $\mathcal{N}^{(0)}$ is the kernel of the action of $\mathbb{C}[\kappa, \kappa^{-1}] \Psi^{(0)}$ on U_0 . By Theorem 3.1 (ii), $\mathbb{C}[k, k^{-1}] \Psi^{(0)} / \mathbb{C}[k, k^{-1}] \Psi^{nil}$ is generated by $k^{\pm 1}$ and $y^i x^i$ ($i = 0, 1, 2, \dots$) mod $\mathbb{C}[k, k^{-1}] \Psi^{nil}$. Apparently $\mathbb{C}[k, k^{-1}] \Psi^{nil}$ belongs to $\mathcal{N}^{(0)}$ and the action of $y^i x^i$ on U_0 is determined by σ_i . Also, using the fact that the \mathcal{T} -module V is type s , the action of $k^{\pm 1}$ on U_0 is determined by s and d . Therefore the action of $\mathbb{C}[k, k^{-1}] \Psi^{(0)}$ on U_0 is determined by s and $\{\sigma_j\}_{j=0}^d$. Since $\mathcal{N}^{(0)} = \mathcal{N} \cap \mathbb{C}[k, k^{-1}] \Psi^{(0)}$ is the kernel of the action, $\mathcal{N}^{(0)}$ is determined by s and $\{\sigma_j\}_{j=0}^d$.

For $i \geq 1$, we claim

$$\mathcal{N}^{(i)} = \{\nu \in \mathbb{C}[k, k^{-1}] \Psi^{(i)} \mid \Psi^{(-i)} \nu \subseteq \mathcal{N}^{(0)}\}.$$

For $\nu \in \mathcal{N}^{(i)} = \mathcal{N} \cap \mathbb{C}[k, k^{-1}] \Psi^{(i)}$, we have $\nu U_0 = 0$ and so $\Psi^{(-i)} \nu U_0 = 0$, i.e., $\Psi^{(-i)} \nu \subseteq \mathcal{N} \cap \mathbb{C}[k, k^{-1}] \Psi^{(0)} = \mathcal{N}^{(0)}$. Conversely, choose $\nu \in \mathbb{C}[k, k^{-1}] \Psi^{(i)}$ such that $\Psi^{(-i)} \nu \subseteq \mathcal{N}^{(0)}$. If $\nu U_0 \neq 0$, then $\mathcal{T} \nu U_0 = V$ by the irreducibility of the \mathcal{T} -module V . Since $\mathcal{T} \nu = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}[k, k^{-1}] \Psi^{(j)} \nu$ and $\Psi^{(j)} \nu U_0 \subseteq U_{j+i}$, we have $\mathcal{T} \nu U_0 = \bigoplus_{j=-i}^{d-i} \Psi^{(j)} \nu U_0$, in particular $\Psi^{(-i)} \nu U_0 = U_0$, which contradicts the assumption $\Psi^{(-i)} \nu \subseteq \mathcal{N}^{(0)}$. Thus $\nu U_0 = 0$, i.e., $\nu \in \mathcal{N} \cap \mathbb{C}[k, k^{-1}] \Psi^{(i)} = \mathcal{N}^{(i)}$, and the claim is proved. This means $\mathcal{N}^{(i)}$ is determined by $\mathcal{N}^{(0)}$. Since $\mathcal{N}^{(0)}$ is determined by s and $\{\sigma_j\}_{j=0}^d$, so is $\mathcal{N}^{(i)}$. This completes the proof of the injectivity of σ in Theorem 1.9. \square

4 Finite-dimensional irreducible \mathcal{A} -modules via ι_t : Proof of Theorem 1.11

The TD-algebra $\mathcal{A} = \mathcal{A}_q^{(\varepsilon, \varepsilon^*)}$ is by Proposition 1.1 embedded into the augmented TD-algebra $\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$ via the injective algebra-homomorphism

$$\iota_t : \mathcal{A} \longrightarrow \mathcal{T} \quad (z \mapsto z_t, z^* \mapsto z_t^*)$$

for each fixed $t \in \mathbb{C}$ ($t \neq 0$), where

$$\begin{aligned} z_t &= x + tk + \varepsilon t^{-1} k^{-1}, \\ z_t^* &= y + \varepsilon^* t^{-1} k + tk^{-1}. \end{aligned}$$

Let V be a finite-dimensional irreducible \mathcal{T} -module of type s and diameter d . As we discussed in Section 1.2, the pair $A = z_t|_V$, $A^* = z_t^*|_V$ of linear transformations of V gives rise to a TD-pair if and only if

- (C₁)_t: the actions of z_t, z_t^* on V are both diagonalizable,
 - (C₂)_t: V is irreducible as a $\langle z_t, z_t^* \rangle$ -module,
- where $\langle z_t, z_t^* \rangle$ is the subalgebra of \mathcal{T} generated by z_t, z_t^* .

By Proposition 1.3, the condition (C₁)_t holds if and only if $\theta_i \neq \theta_j$ and $\theta_i^* \neq \theta_j^*$ for $i \neq j$ ($0 \leq i, j \leq d$), where

$$\begin{aligned} \theta_i &= s t q^{2i-d} + \varepsilon s^{-1} t^{-1} q^{d-2i}, \\ \theta_i^* &= \varepsilon^* s t^{-1} q^{2i-d} + s^{-1} t q^{d-2i}. \end{aligned}$$

In this section, we prove Theorem 1.11, a criterion for (C₂)_t. Namely assume (C₁)_t. Then the condition (C₂)_t holds if and only if $P_V(t^2 + \varepsilon \varepsilon^* t^{-2}) \neq 0$, where $P_V(\lambda)$ is the Drinfel'd polynomial of the \mathcal{T} -module V .

We proceed parallel to [7]. Let $V = \bigoplus_{i=0}^d U_i$ denote the weight-space decomposition of the \mathcal{T} -module V , and F_i the projection of $V = \bigoplus_{i=0}^d U_i$ onto U_i . Note that k acts on V as $\sum_{i=0}^d s q^{2i-d} F_i$. Identifying z, z^* with z_t, z_t^* via ι_t , we write $z = z_t, z^* = z_t^*$ for short. Since (C₁)_t is assumed, the action of z (resp. z^*) on V has $d+1$ distinct eigenvalues $\theta_0, \dots, \theta_d$ (resp. $\theta_0^*, \dots, \theta_d^*$) on V by Proposition 1.3. Let V_i (resp. V_i^*) denote the eigenspace of z (resp. z^*) on V belonging to θ_i (resp. θ_i^*). Then we have

$$\begin{aligned} V_i + V_{i+1} + \dots + V_d &= U_i + U_{i+1} + \dots + U_d, \\ V_0^* + V_1^* + \dots + V_i^* &= U_0 + U_1 + \dots + U_i \end{aligned}$$

for $0 \leq i \leq d$. In particular, $U_0 = V_0^*, U_d = V_d$. Let E_i (resp. E_i^*) denote the projection of $V = \bigoplus_{i=0}^d V_i$ (resp. $V = \bigoplus_{i=0}^d V_i^*$) onto V_i (resp. V_i^*). Then the mappings

$$\begin{aligned} F_i|_{V_i} : V_i &\longrightarrow U_i, \\ E_i|_{U_i} : U_i &\longrightarrow V_i \end{aligned}$$

are both bijections and inverses each other. Also the mappings

$$\begin{aligned} F_i|_{V_i^*} : V_i^* &\longrightarrow U_i, \\ E_i^*|_{U_i} : U_i &\longrightarrow V_i^* \end{aligned}$$

are both bijections and inverses each other. In particular, by Theorem 1.8

$$\dim V_0 = 1, \quad \dim V_d^* = 1.$$

By the argument in the proof of Proposition 1.7, the TD-relations (TD) for z, z^* imply

$$\begin{aligned} z^* V_i &\subseteq V_{i-1} + V_i + V_{i+1}, \\ z V_i^* &\subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \end{aligned}$$

for $0 \leq i \leq d$, where $V_{-1} = V_{d+1} = V_{-1}^* = V_{d+1}^* = 0$.

Regard V as an \mathcal{A} -module via ι_t . Let W be an irreducible \mathcal{A} -submodule of V . Set $W_i = W \cap V_i$, $W_i^* = W \cap V_i^*$. Then

$$\begin{aligned} z^* W_i &\subseteq W_{i-1} + W_i + W_{i+1}, \\ z W_i^* &\subseteq W_{i-1}^* + W_i^* + W_{i+1}^* \end{aligned}$$

for $0 \leq i \leq d$. Since W is irreducible as an \mathcal{A} -module and since z, z^* are diagonalizable on W , the pair $z|_W, z^*|_W$ is a TD-pair on W . This implies that the eigenspace decompositions of $z|_W, z|_{W^*}$ are

$$\begin{aligned} W &= \bigoplus_{i=r}^{r+d'} W_i, \\ W &= \bigoplus_{i=r'}^{r'+d'} W_i^*, \end{aligned}$$

for some integers r, r' , where d' is the diameter of the TD-pair $z|_W, z^*|_W \in \text{End}(W)$. As we discussed in Section 1.2, the \mathcal{A} -module structure on W can be extended to a \mathcal{T} -module structure on W by using the split decomposition of the TD-pair $z|_W, z^*|_W$. (Note that the weight-space decomposition of the \mathcal{T} -module W may be totally different from that of the \mathcal{T} -module V .) By applying Theorem 1.8 to the irreducible \mathcal{T} -module W , we have

$$\dim W_r = \dim W_{r'+d'}^* = 1.$$

First we want to show $r = 0, r' + d' = d$, i.e., $W \supseteq V_0, W \supseteq V_d^*$.

Since $\dim W_r = 1$, we have $W_r = \mathbb{C}v$ for some nonzero element $v \in W_r$. Since $W \subseteq V_r + \cdots + V_d = U_r + \cdots + U_d$, we can express v as

$$v = u_r + \cdots + u_d,$$

where $u_i = F_i v \in U_i$. Then $u_r \neq 0$, since $v \in W_r \subseteq V_r$ and $F_r|_{V_r} : V_r \rightarrow U_r$ is a bijection.

Lemma 4.1 *The action of \mathcal{T} on V satisfies the following (i), (ii), (iii).*

- (i) $x^j u_r = (\theta_r - \theta_{r+1}) \cdots (\theta_r - \theta_{r+j}) u_{r+j} \quad (1 \leq j \leq d - r).$
- (ii) $y u_r = 0.$
- (iii) $y^j u_{r+j} \in \mathbb{C} u_r \quad (1 \leq j \leq d - r).$

Proof. Recall $z = z_t$ and so $z|_V = x|_V + \sum_{i=0}^d \theta_i F_i$. Since $u_i \in U_i$, we have $z u_i = x u_i + \theta_i u_i$, so $z v = \theta_r u_r + (x u_r + \theta_{r+1} u_{r+1}) + \cdots + (x u_d + \theta_d u_d)$. Note $x u_i \in U_{i+1}$. On the other hand, since $v \in W_r \subseteq V_r$, we have $z v = \theta_r v = \theta_r u_r + \theta_r u_{r+1} + \cdots + \theta_r u_d$. Therefore we have $x u_{i-1} + \theta_i u_i = \theta_r u_i$, i.e., $x u_{i-1} = (\theta_r - \theta_i) u_i$ and we obtain (i) recursively.

Recall $z^* = z_t^*$ and so $z^*|_V = y|_V + \sum_{i=0}^d \theta_i^* F_i$. Since $u_i \in U_i$, we have $z^* u_i = y u_i + \theta_i^* u_i$, so $z^* v = y u_r + (y u_{r+1} + \theta_{r+1}^* u_{r+1}) + \cdots + (y u_d + \theta_d^* u_d)$. Note $y u_i \in U_{i-1}$. On the other hand, since $z^* v \in W$ and $F_{r-1} W = 0$, we have $y u_r = 0$, i.e., (ii) holds. Since $z^* v \in W$ and $F_r W = F_r W_r = \mathbb{C} u_r$, we have $y u_{r+1} + \theta_{r+1}^* u_r \in \mathbb{C} u_r$, i.e., (iii) holds for $j = 1$. By $z^*|_V = y|_V + \sum_{i=0}^d \theta_i^* F_i$ and $y U_i \subseteq U_{i-1}$, we can write $z^{*j} u_i$ as a linear combination of $u_i, y u_i, y^2 u_i, \dots, y^j u_i$, in which the coefficient of $y^j u_i$ is 1 if $i - j \geq r$. In particular for $v = u_r + \cdots + u_d$, the projection of $z^{*j} v$ onto U_r by F_r can be written as

$$F_r z^{*j} v = y^j u_{r+j} + c_1 y^{j-1} u_{r+j-1} + \cdots + c_{j-1} y u_{r+1} + c_j u_r$$

for some $c_1, \dots, c_{j-1}, c_j \in \mathbb{C}$. Since $F_r z^{*j} v \in F_r W = F_r W_r = \mathbb{C} u_r$, (iii) holds by induction on j . \square

Proposition 4.2 *It holds that $W \supseteq V_0$ and $W \supseteq V_d^*$.*

Proof. We only show $W \supseteq V_0$, i.e., $r = 0$; $W \supseteq V_d^*$ is proved similarly, using $W_{r'+d'}^*$ in place of W_r . By Lemma 4.1, the action of \mathcal{T} on V satisfies

$$\begin{aligned} y u_r &= 0, \\ y^j x^j u_r &\in \mathbb{C} u_r \quad (j = 0, 1, 2, \dots). \end{aligned}$$

This implies $\mathcal{T} u_r \subseteq U_r + \cdots + U_d$, since \mathcal{T} is linealy spanned by $k^n w_\lambda(x, y)$ ($n \in \mathbb{Z}, \lambda \in \Lambda^{irr}$) by Theorem 2.2. Since V is irreducible as a \mathcal{T} -module, we have $V = \mathcal{T} u_r$ and hence $r = 0$. \square

Thus for a finite-dimensional irreducible \mathcal{T} -module V of type s , diameter d and an irreducible \mathcal{A} -module $W \subseteq V$ via ι_t , we have $W_0 = V_0$, $W_d^* = V_d^*$. In particular, $W = \mathcal{A} V_0$.

Next we calculate how the eigenspace V_0 of $z|_V$ is mapped to V_0^* by the projection $E_0^* : V = \bigoplus_{i=0}^d V_i^* \longrightarrow V_0^*$. It holds that on V

$$E_0^* = \prod_{j=1}^d \frac{z^* - \theta_j^*}{\theta_0^* - \theta_j^*},$$

since the right-hand side vanishes on $V_j^* (1 \leq j \leq d)$ and is the identity map on V_0^* . Write V_0 as $V_0 = \mathbb{C} v$ for some nonzero element $v \in V_0$ and express v as $v = u_0 + u_1 + \cdots + u_d$, where $u_i = F_i v \in U_i$. Then we obtain

$$E_0^* u_i = \Theta_i^{*-1} y^i u_i, \quad \Theta_i^* = \prod_{j=1}^i (\theta_0^* - \theta_j^*).$$

This is because $(\prod_{j=1}^i (z^* - \theta_j^*)) u_i = y^i u_i \in U_0 = V_0^*$ by $(z^* - \theta_j^*)|_{U_j} = y|_{U_j}$, $y U_j \subseteq U_{j-1}$ and because $(z^* - \theta_j^*)|_{U_0} = \theta_0^* - \theta_j^*$ for $i+1 \leq j \leq d$. By Lemma 4.1 with $r = 0$,

$$u_i = \Theta_i^{-1} x^i u_0, \quad \Theta_i = \prod_{j=1}^i (\theta_0 - \theta_j).$$

Since $y^i x^i u_0 = \sigma_i u_0$, we have

$$E_0^* u_i = \Theta_i^{-1} \Theta_i^{*-1} \sigma_i u_0$$

and so

$$E_0^* v = \sum_{i=0}^d \Theta_i^{-1} \Theta_i^{*-1} \sigma_i u_0.$$

Note that $u_0 = F_0 v \neq 0$, since $F_0|_{V_0} : V_0 \rightarrow U_0$ is a bijection. Thus by Remark 1.10 in Section 1.3, we have

Proposition 4.3 *For a finite-dimensional irreducible \mathcal{T} -module V of type s and diameter d , assume the condition $(C_1)_t$ for a nonzero $t \in \mathbb{C}$. Then for $v \in V_0$, it holds that*

$$E_0^* v = \Theta^{-1} Q P_V(t^2 + \varepsilon \varepsilon^* t^{-2}) u_0,$$

where $u_0 = F_0 v$,

$$\begin{aligned} \Theta &= (\theta_0 - \theta_1) \cdots (\theta_0 - \theta_d)(\theta_0^* - \theta_1^*) \cdots (\theta_0^* - \theta_d^*), \\ Q &= (-1)^d (q - q^{-1})^2 (q^2 - q^{-2})^2 \cdots (q^d - q^{-d})^2, \end{aligned}$$

and $P_V(\lambda)$ is the Drinfel'd polynomial of the \mathcal{T} -module V defined in Section 1.3:

$$P_V(\lambda) = Q^{-1} \sum_{i=0}^d \sigma_i(V) \prod_{j=i+1}^d (q^j - q^{-j})^2 (\varepsilon s^{-2} q^{2(d-j)} + \varepsilon^* s^2 q^{-2(d-j)} - \lambda).$$

Proof of Theorem 1.11. Suppose $P_V(t^2 + \varepsilon \varepsilon^* t^{-2}) \neq 0$. Then by Proposition 4.3, we have $E_0^* V_0 \neq 0$. Then $E_0^* V_0 = V_0^*$, since $E_0^* V_0 \subseteq V_0^*$ and $\dim V_0^* = 1$. Let W be an irreducible \mathcal{A} -submodule of V via ι_t . Then $W \supseteq V_0$ by Proposition 4.2. Since E_0^* is a polynomial of $z^*|_V$, W is E_0^* -invariant and so $W \supseteq E_0^* W \supseteq E_0^* V_0 = V_0^*$, i.e., $W \supseteq U_0$ by $U_0 = V_0^*$. We want to prove $W = V$. To do so, it is enough to show $\omega U_0 \subseteq W$ for every word ω in x, y , since $V = \mathcal{T} U_0$ and $\mathcal{T} U_0$ is linearly spanned by such ωU_0 's. Now ωU_0 belongs to some U_i and x, y coincide with $z - \theta_i$, $z^* - \theta_i^*$ on U_i respectively. Therefore $\omega U_0 \subseteq W$ implies $x \omega U_0 \subseteq W$ and $y \omega U_0 \subseteq W$, since W is invariant under $z - \theta_i$, $z^* - \theta_i^*$. This means induction works on the word length. Thus $\omega U_0 \subseteq W$ holds for every word ω in x, y and it is shown that $P_V(t^2 + \varepsilon \varepsilon^* t^{-2}) \neq 0$ implies $W = V$, i.e., V is irreducible as an \mathcal{A} -module.

Suppose $P_V(t^2 + \varepsilon \varepsilon^* t^{-2}) = 0$. Then by Proposition 4.3, we have $E_0^* V_0 = 0$. This means $V_0 \subseteq V_1^* + \cdots + V_d^*$. Set

$$V_{i,i+1} = (V_0 + \cdots + V_i) \cap (V_{i+1}^* + \cdots + V_d^*)$$

for $0 \leq i \leq d-1$. Note $V_0 = V_{0,1}$. Then by $zV_j^* \subseteq V_{j-1}^* + V_j^* + V_{j+1}^*$ and $z^*V_j \subseteq V_{j-1} + V_j + V_{j+1}$, we have

$$\begin{aligned} (z - \theta_i)V_{i,i+1} &\subseteq V_{i-1,i}, \\ (z^* - \theta_{i+1}^*)V_{i,i+1} &\subseteq V_{i+1,i+2}, \end{aligned}$$

where $V_{-1,0} = V_{d,d+1} = 0$. Set $V' = V_{0,1} + V_{1,2} + \cdots + V_{d-1,d}$. Then V' is $\langle z, z^* \rangle$ -invariant. Since $V_0 \subseteq V' \subseteq V_1^* + \cdots + V_d^*$, the $\langle z, z^* \rangle$ -invariant subspace V' is a proper subspace of V . Thus it is shown that if $P_V(t^2 + \varepsilon \varepsilon^* t^{-2}) = 0$, then V is not irreducible as an \mathcal{A} -module. This completes the proof of Theorem 1.11. \square

5 The product formula for the Drinfel'd polynomial $P_V(\lambda)$ of a \mathcal{T} -module V via φ_s : Proof of the surjectivity of σ in Theorem 1.9

The augmented TD-algebra $\mathcal{T} = \mathcal{T}_q^{\varepsilon, \varepsilon^*}$ is by Proposition 1.13 embedded into the $U_q(sl_2)$ -loop algebra $\mathcal{L} = U_q(L(sl_2))$ via the injective algebra-homomorphism

$$\varphi_s : \mathcal{T} \longrightarrow \mathcal{L} \quad (x, y, k \mapsto x(s), y(s), sk_0 \text{ respectively})$$

for each fixed nonzero $s \in \mathbb{C}$, where

$$\begin{aligned} x(s) &= \alpha(se_0^+ + \varepsilon s^{-1}e_1^-k_1), & \alpha &= -q^{-1}(q - q^{-1})^2, \\ y(s) &= \varepsilon^*se_0^-k_0 + s^{-1}e_1^+. \end{aligned}$$

For $(\varepsilon, \varepsilon^*) = (1, 1)$, $(0, 0)$, let

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

be the tensor product of evaluation modules $V(\ell_i, a_i)$ for \mathcal{L} ($1 \leq \ell_i, a_i \in \mathbb{C} \setminus \{0\}$, $1 \leq i \leq n$) (see Section 1.4). We regard V as a \mathcal{T} -module via the embedding φ_s . We call such a \mathcal{T} -module V a *tensor product of evaluation modules via φ_s* .

For $(\varepsilon, \varepsilon^*) = (1, 0)$, let \mathcal{L}' denote the subalgebra of \mathcal{L} generated by $e_0^+, e_1^+, e_1^-, k_i^{\pm 1}$ ($i = 0, 1$) and $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ the tensor product of evaluation modules $V(\ell_i, a_i)$ for \mathcal{L}' ($1 \leq \ell_i, a_i \in \mathbb{C}$, $1 \leq i \leq n$): note that e_0^- is missing from the set of generators for \mathcal{L}' and $a_i = 0$ is allowed for the evaluation module $V(\ell_i, a_i)$ of \mathcal{L}' (see Section 1.4). We regard V as a \mathcal{T} -module via the embedding φ_s , since the image of \mathcal{T} by φ_s is contained in \mathcal{L}' in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$. We call such a \mathcal{T} -module V a *tensor product of evaluation modules via φ_s* .

We treat such a \mathcal{T} -module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s in one argument, regardless of $(\varepsilon, \varepsilon^*)$, and use the same notation \mathcal{L} for \mathcal{L}' in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$. So in this section, we understand in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$ that \mathcal{L} denotes the subalgebra of the $U_q(sl_2)$ -loop algebra $U_q(L(sl_2))$ generated by $e_0^+, e_1^+, e_1^-, k_i^{\pm 1}$ ($i = 0, 1$) with e_0^- missing from the set of generators, and that $a_i = 0$ is allowed for the evaluation module $V(\ell_i, a_i)$.

For a \mathcal{T} -module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via φ_s , let $v_0^{(i)}, \dots, v_{\ell_i}^{(i)}$ denote a standard basis of $V(\ell_i, a_i)$: we write $v_0 = v_0^{(i)}, v_1 = v_1^{(i)}, \dots, v_{\ell_i} = v_{\ell_i}^{(i)}$ for short. The action of \mathcal{T} on $V(\ell_i, a_i) = \langle v_0, v_1, \dots, v_{\ell_i} \rangle$ is

$$\begin{aligned} k_0 v_j &= q^{2j-\ell_i} v_j, \\ e_0^+ v_j &= a_i q [j+1] v_{j+1}, \\ \varepsilon^* e_0^- v_j &= \varepsilon^* a_i^{-1} q^{-1} [\ell_i - j + 1] v_{j-1}, \\ e_1^+ v_j &= [\ell_i - j + 1] v_{j-1}, \\ e_1^- v_j &= [j+1] v_{j+1}, \end{aligned}$$

where $v_{-1}^{(i)} = v_{\ell_i+1}^{(i)} = 0$, and we understand $\varepsilon^* a_i^{-1} = 0$ if $(\varepsilon, \varepsilon^*) = (1, 0)$ and $a_i = 0$. Let U_i denote the subspace of V spanned by $v_{j_1} \otimes \cdots \otimes v_{j_n}$, where (j_1, \dots, j_n) runs through $0 \leq j_1 \leq \ell_1, \dots, 0 \leq j_n \leq \ell_n$ such that $j_1 + \cdots + j_n = i$:

$$U_i = \bigoplus_{j_1 + \cdots + j_n = i} \mathbb{C} v_{j_1} \otimes \cdots \otimes v_{j_n}.$$

Then $k|_{U_i} = sq^{2i-d}$, so

$$V = \bigoplus_{i=0}^d U_i \quad (d = \ell_1 + \cdots + \ell_n)$$

is the eigenspace decomposition of $\varphi_s(k)$. We call $V = \bigoplus_{i=0}^d U_i$ the *weight space decomposition* of the \mathcal{T} -module V via φ_s and U_0 the *highest weight space*. Observe that

$$\begin{aligned} \dim U_0 &= 1, \\ x U_i &\subseteq U_{i+1}, \quad y U_i \subseteq U_{i-1} \end{aligned}$$

for $0 \leq i \leq d$, where $U_{-1} = U_{d+1} = 0$. So the 1-dimensional space U_0 is invariant under $y^i x^i$. Define the sequence $\{\sigma_i\}_{i=0}^\infty$ of scalars $\sigma_i = \sigma_i(V)$ by

$$y^i x^i|_{U_0} = \sigma_i.$$

Then $\sigma_0 = 1, \sigma_i = 0$ ($d+1 \leq i$). Note that the \mathcal{T} -module V via φ_s is not necessarily irreducible and $\sigma_d = 0$ is possible. Define the Drinfel'd polynomial $P_V(\lambda)$ of the \mathcal{T} -module V via φ_s by

$$P_V(\lambda) = Q^{-1} \sum_{i=0}^d \sigma_i(V) \prod_{j=i+1}^d (q^j - q^{-j})^2 (\varepsilon s^{-2} q^{2(d-j)} + \varepsilon^* s^2 q^{-2(d-j)} - \lambda), \quad (25)$$

$$Q = Q_d = (-1)^d (q - q^{-1})^2 (q^2 - q^{-2})^2 \cdots (q^d - q^{-d})^2. \quad (26)$$

Since $\sigma_0 = 1$, $P_V(\lambda)$ is a monic polynomial of degree d . Observe

$$\sigma_d(V) = Q \cdot P_V(\varepsilon s^{-2} + \varepsilon^* s^2).$$

More generally the Drinfel'd polynomial $P_V(\lambda)$ is defined in the same way for a finite-dimensional \mathcal{T} -module V that has the following properties:

- (D)₀: k is diagonalizable on V with $V = \bigoplus_{i=0}^d U_i$, $k|_{U_i} = sq^{2i-d}$ ($0 \leq i \leq d$)
for some nonzero constant s .
(D)₁: $\dim U_0 = 1$.

By the relations (TD)₀ : $kk^{-1} = k^{-1}k = 1$, $kxk^{-1} = q^2x$, $kyk^{-1} = q^{-2}y$, it holds that $xU_i \subseteq U_{i+1}$, $yU_i \subseteq U_{i-1}$ ($0 \leq i \leq d$), where $U_{-1} = U_{d+1} = 0$. Thus $\sigma_i(V)$'s are defined as before and hence $P_V(\lambda)$ by (25), (26). The eigenspace decomposition and the subspace U_0 in (D)₀ are called the *weight-space decomposition* and the *highest weight space* of the \mathcal{T} -module V respectively. The nonzero scalar s and the nonnegative integer d in (D)₀ are called the *type* and the *diameter* of the \mathcal{T} -module V respectively. We further consider the following property for a \mathcal{T} -module V that satisfies (D)₀, (D)₁ with diameter d :

- (D)₂: $\sigma_d(V) \neq 0$.

Lemma 5.1 *Let V be a finite-dimensional \mathcal{T} -module that satisfies the properties (D)₀, (D)₁. Consider the \mathcal{T} -submodule $W = \mathcal{T}U_0$, where U_0 is the highest weight space of the \mathcal{T} -module V . Let M be a maximal \mathcal{T} -submodule of W . Set $\overline{W} = W/M$. Then the \mathcal{T} -submodule W and the quotient \mathcal{T} -module \overline{W} satisfy (D)₀, (D)₁ as well. Furthermore if V satisfies (D)₂ with diameter d , then so do the \mathcal{T} -modules W and \overline{W} and it holds that*

- (i) $\sigma_i(V) = \sigma_i(W) = \sigma_i(\overline{W})$ ($0 \leq i \leq d$),
(ii) $P_V(\lambda) = P_W(\lambda) = P_{\overline{W}}(\lambda)$.

Lemma 5.1 follows from Lemma 1.2, since \overline{W} is irreducible as a \mathcal{T} -module.

In what follows, we fix a nonzero scalar $s \in \mathbb{C}$ arbitrarily and we only treat finite-dimensional \mathcal{T} -modules via φ_s that satisfy the above properties (D)₀, (D)₁. In this case, the weight space decomposition of a \mathcal{T} -module V coincides with that of the \mathcal{L} -module V , since $\varphi_s(k) = sk_0$. Note that the tensor product of evaluation modules $V(\ell_i, a_i)$ ($1 \leq i \leq n$) satisfies (D)₀, (D)₁ and has type s , diameter $d = \ell_1 + \cdots + \ell_n$. If V, V' are \mathcal{T} -modules via φ_s , then the tensor product $V \otimes V'$ becomes a \mathcal{T} -module via $\Delta \circ \varphi_s$, where $\Delta : \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{L}$ is the coproduct. Furthermore, if the \mathcal{T} -modules V, V' via φ_s satisfy the properties (D)₀, (D)₁, so does the tensor product $V \otimes V'$ as a \mathcal{T} -module via φ_s and so the Drinfel'd polynomial $P_{V \otimes V'}(\lambda)$ is defined. We have the following product formula.

Theorem 5.2 *Let V, V' be finite-dimensional \mathcal{T} -modules via φ_s that satisfy the properties (D)₀, (D)₁. Assume that V' is afforded by a tensor product of evaluation modules via φ_s . Then the following (i), (ii) holds.*

- (i) *The Drinfel'd polynomial $P_{V \otimes V'}(\lambda)$ of the \mathcal{T} -module $V \otimes V'$ via φ_s is*

$$P_{V \otimes V'}(\lambda) = P_V(\lambda)P_{V'}(\lambda).$$

- (ii) *The Drinfel'd polynomial $P_{V(\ell, a)}(\lambda)$ of the \mathcal{T} -module $V(\ell, a)$ via φ_s is*

$$P_{V(\ell, a)}(\lambda) = \prod_{c \in S(\ell, a)} (\lambda + c + \varepsilon \varepsilon^* c^{-1}),$$

where

$$S(\ell, a) = \{a q^{2i-\ell+1} \mid 0 \leq i \leq \ell - 1\}.$$

We understand that if $(\varepsilon, \varepsilon^*) = (1, 0)$ and $a = 0$, $S(\ell, a)$ is the multiset with 0 appearing ℓ times and $P_{V(\ell, 0)} = \lambda^\ell$.

To prove Theorem 5.2, we prepare two lemmas and a proposition. Let V, V' be \mathcal{T} -modules via φ_s as in Theorem 5.2 and have weight-space decompositions

$$\begin{aligned} V &= \bigoplus_{i=0}^d U_i, \\ V' &= \bigoplus_{i=0}^{d'} U'_i, \end{aligned}$$

respectively. Then the \mathcal{T} -module $V \otimes V'$ via φ_s has weight-space decomposition

$$V \otimes V' = \bigoplus_{i=0}^{d+d'} \tilde{U}_i,$$

where

$$\tilde{U}_i = \bigoplus_{i_1+i_2=i} U_{i_1} \otimes U_{i_2} \quad (0 \leq i \leq d+d').$$

Lemma 5.3 *Set $x(s) = \varphi_s(x)$, $y(s) = \varphi_s(y)$. Then the actions of $x(s)$, $y(s)$ on $U_i \otimes V'$ are*

$$\begin{aligned} x(s)|_{U_i \otimes V'} &= x(s)|_{U_i} \otimes 1_{V'} + 1_{U_i} \otimes x(q^{2i-d}s)|_{V'}, \\ y(s)|_{U_i \otimes V'} &= y(s)|_{U_i} \otimes 1_{V'} + 1_{U_i} \otimes y(q^{2i-d}s)|_{V'}. \end{aligned}$$

Proof. These identities follow directly from $x(s) = \alpha(se_0^+ + \varepsilon s^{-1}e_1^-k_1)$, $y(s) = \varepsilon^*se_0^-k_0 + s^{-1}e_1^+$ and the coproduct Δ that sends e_i^+ , $e_i^-k_i$, k_i to $e_i^+ \otimes 1 + k_i \otimes e_i^+$, $e_i^-k_i \otimes 1 + k_i \otimes e_i^-k_i$, $k_i \otimes k_i$ respectively. \square

Lemma 5.4 *Assume $V' = V(1, a)$, an evaluation module of diameter 1, and let $V(1, a) = \langle v_0, v_1 \rangle$ be a standard basis. For $u \in U_m$ ($0 \leq m \leq d$) and $1 \leq i$, we have*

$$\begin{aligned} (i) \quad x^i(u \otimes v_0) &= (x^i u) \otimes v_0 + \alpha q[i] c_i(m) (x^{i-1} u) \otimes v_1, \\ x^i(u \otimes v_1) &= (x^i u) \otimes v_1, \end{aligned}$$

$$\text{where } c_i(m) = a s q^{i+2m-d-1} + \varepsilon s^{-1} q^{-i-2m+d+1},$$

$$\begin{aligned} (ii) \quad y^i(u \otimes v_0) &= (y^i u) \otimes v_0, \\ y^i(u \otimes v_1) &= (y^i u) \otimes v_1 + [i] c_i^*(m) (y^{i-1} u) \otimes v_0, \end{aligned}$$

$$\text{where } c_i^*(m) = \varepsilon^* a^{-1} s q^{-i+2m-d+1} + s^{-1} q^{i-2m+d-1}.$$

Proof. Recall $e_0^+ v_0 = q a v_1$, $e_1^- v_0 = v_1$, $e_0^+ v_1 = e_1^- v_1 = 0$, $\varepsilon^* e_0^- v_1 = \varepsilon^* a^{-1} q^{-1} v_0$, $e_1^+ v_1 = v_0$, $\varepsilon^* e_0^- v_0 = e_1^+ v_0 = 0$, $k_0 v_0 = q^{-1} v_0$, $k_0 v_1 = q v_1$. We proceed by induction on i . For $i = 1$, we have by Lemma 5.3

$$\begin{aligned} x(u \otimes v_0) &= (x(s)u) \otimes v_0 + u \otimes (x(q^{2m-d}s) v_0) \\ &= (xu) \otimes v_0 + \alpha (q^{2m-d}s q a + \varepsilon q^{-2m+d}s^{-1} q) u \otimes v_1 \\ &= (xu) \otimes v_0 + \alpha q c_1(m) u \otimes v_1, \\ x(u \otimes v_1) &= (x(s)u) \otimes v_1 + u \otimes (x(q^{2m-d}s) v_1) \\ &= (xu) \otimes v_1, \end{aligned}$$

and

$$\begin{aligned} y(u \otimes v_0) &= (y(s)u) \otimes v_0 + u \otimes (y(q^{2m-d}s) v_0) \\ &= (yu) \otimes v_0, \\ y(u \otimes v_1) &= (y(s)u) \otimes v_1 + u \otimes (y(q^{2m-d}s) v_1) \\ &= (yu) \otimes v_1 + (\varepsilon^* q^{2m-d}s a^{-1} + q^{-2m+d}s^{-1}) u \otimes v_0 \\ &= (yu) \otimes v_1 + c_1^*(m) u \otimes v_0. \end{aligned}$$

For $i \geq 2$, we have by Lemma 5.3 and induction on i

$$\begin{aligned} x^i(u \otimes v_0) &= x^{i-1}((xu) \otimes v_0 + \alpha q c_1(m) u \otimes v_1) \\ &= (x^i u) \otimes v_0 + \alpha q [i-1] c_{i-1}(m+1) (x^{i-1} u) \otimes v_1 \\ &\quad + \alpha q c_1(m) (x^{i-1} u) \otimes v_1 \\ &= (x^i u) \otimes v_0 + \alpha q [i] c_i(m) (x^{i-1} u) \otimes v_1, \end{aligned}$$

since $[i-1]c_{i-1}(m+1) + c_1(m) = [i]c_i(m)$, and

$$\begin{aligned} y^i(u \otimes v_1) &= y^{i-1}((yu) \otimes v_1 + c_1^*(m) u \otimes v_0) \\ &= (y^i u) \otimes v_1 + [i-1] c_{i-1}^*(m-1) (y^{i-1} u) \otimes v_0 \\ &\quad + c_1^*(m) (y^{i-1} u) \otimes v_0 \\ &= (y^i u) \otimes v_1 + [i] c_i^*(m) (y^{i-1} u) \otimes v_0, \end{aligned}$$

since $[i-1]c_{i-1}^*(m-1) + c_1^*(m) = [i]c_i^*(m)$. Also we have $x^i(u \otimes v_1) = x^{i-1}((xu) \otimes v_1) = (x^i u) \otimes v_1$, $y^i(u \otimes v_0) = y^{i-1}((yu) \otimes v_0) = (y^i u) \otimes v_0$ by induction on i . \square

Proposition 5.5 *Assume $V' = V(1, a)$, an evaluation module of diameter 1. For the \mathcal{T} -modules V and $V \otimes V'$ via φ_s , set $\sigma_i = \sigma_i(V)$ and $\tilde{\sigma}_i = \sigma_i(V \otimes V')$. Then for $i \geq 1$, we have*

$$\tilde{\sigma}_i = \sigma_i - (q^i - q^{-i})^2 (a + \varepsilon \varepsilon^* a^{-1} + \varepsilon s^{-2} q^{2(d+1-i)} + \varepsilon^* s^2 q^{-2(d+1-i)}) \sigma_{i-1},$$

where d is the diameter of the \mathcal{T} -module V . We understand that $\varepsilon^* a^{-1} = 0$ if $(\varepsilon, \varepsilon^*) = (1, 0)$ and $a = 0$.

Proof. Let $V = \bigoplus_{i=0}^d U_i$ denote the weight-space decomposition of the \mathcal{T} -module V and $V(1, a) = \langle v_0, v_1 \rangle$ a standard basis of V' . Choose a nonzero vector $u_0 \in U_0$. Then $u_0 \otimes v_0$ spans the highest weight space of $V \otimes V'$. We have by Lemma 5.4

$$\begin{aligned} y^i x^i (u_0 \otimes v_0) &= y^i \left((x^i u_0) \otimes v_0 + \alpha q [i] c_i(0) (x^{i-1} u_0) \otimes v_1 \right) \\ &= (y^i x^i u_0) \otimes v_0 \\ &\quad + \alpha q [i] c_i(0) \left((y^i x^{i-1} u_0) \otimes v_1 + [i] c_i^*(i-1) (y^{i-1} x^{i-1} u_0) \otimes v_0 \right) \\ &= \sigma_i u_0 \otimes v_0 + \alpha q [i]^2 c_i(0) c_i^*(i-1) \sigma_{i-1} u_0 \otimes v_0, \end{aligned}$$

since $y^i x^{i-1} u_0 = 0$. So it holds that

$$\begin{aligned} \tilde{\sigma}_i &= \sigma_i + \alpha q [i]^2 c_i(0) c_i^*(i-1) \sigma_{i-1} \\ &= \sigma_i - (q^i - q^{-i})^2 (a + \varepsilon \varepsilon^* a^{-1} + \varepsilon s^{-2} q^{2(d+1-i)} + \varepsilon^* s^2 q^{-2(d+1-i)}) \sigma_{i-1}. \end{aligned}$$

□

Proof of Theorem 5.2. We first treat the case of $V' = V(1, a)$, an evaluation module of diameter 1. By Proposition 5.5,

$$\tilde{\sigma}_i = \sigma_i - (q^i - q^{-i})^2 \sigma_{i-1} \left((\varepsilon s^{-2} q^{2(d+1-i)} + \varepsilon^* s^2 q^{-2(d+1-i)} - \lambda) + (\lambda + a + \varepsilon \varepsilon^* a^{-1}) \right),$$

and we have, with $\tilde{Q} = (-1)^{d+1} (q - q^{-1})^2 (q^2 - q^{-2})^2 \cdots (q^{d+1} - q^{-d-1})^2$,

$$\begin{aligned} P_{V \otimes V(1, a)}(\lambda) &= \tilde{Q}^{-1} \sum_{i=0}^{d+1} \tilde{\sigma}_i \prod_{j=i+1}^{d+1} (q^j - q^{-j})^2 (\varepsilon s^{-2} q^{2(d+1-j)} + \varepsilon^* s^2 q^{-2(d+1-j)} - \lambda) \\ &= \tilde{Q}^{-1} \sum_{i=0}^{d+1} \sigma_i \prod_{j=i+1}^{d+1} (q^j - q^{-j})^2 (\varepsilon s^{-2} q^{2(d+1-j)} + \varepsilon^* s^2 q^{-2(d+1-j)} - \lambda) \\ &\quad - \tilde{Q}^{-1} \sum_{i=1=0}^d \sigma_{i-1} \prod_{j=i}^{d+1} (q^j - q^{-j})^2 (\varepsilon s^{-2} q^{2(d+1-j)} + \varepsilon^* s^2 q^{-2(d+1-j)} - \lambda) \\ &\quad - \tilde{Q}^{-1} \sum_{i=1=0}^d \sigma_{i-1} (\lambda + a + \varepsilon \varepsilon^* a^{-1}) (q^{d+1} - q^{-d-1})^2 \\ &\quad \times \prod_{j=1=i}^d (q^{j-1} - q^{-j+1})^2 (\varepsilon s^{-2} q^{2(d-j+1)} + \varepsilon^* s^2 q^{-2(d-j+1)} - \lambda). \end{aligned}$$

This equals $(\lambda + a + \varepsilon \varepsilon^* a^{-1}) P_V(\lambda)$, since $\sigma_{d+1} = \sigma_{d+1}(V) = 0$ and so the first and second terms cancel out. This argument is valid even if V is the trivial module, i.e., $\dim V = 1$, $e_i^\pm|_V = 0$, $k_i^{\pm 1}|_V = 1$. In this case, $V \otimes V(1, a) \simeq V(1, a)$ and it is easily checked that $P_V(\lambda) = 1$ and

$$P_{V(1, a)}(\lambda) = \lambda + a + \varepsilon \varepsilon^* a^{-1}. \quad (27)$$

Thus in the case of $V' = V(1, a)$, we have

$$P_{V \otimes V(1, a)}(\lambda) = P_V(\lambda) P_{V(1, a)}(\lambda). \quad (28)$$

Next we treat the case $V' = V(\ell, a)$, an evaluation module of diameter ℓ . We want to show

$$P_{V \otimes V(\ell, a)}(\lambda) = P_V(\lambda) P_{V(\ell, a)}(\lambda) \quad (29)$$

for every integer $\ell \geq 1$ by induction on ℓ . To do so, we prepare a lemma below that gives an embedding of $V' = V(\ell, a)$ into $V(\ell - 1, a q^{-1}) \otimes V(1, a q^{\ell-1})$ as an \mathcal{L} -submodule. Start with the evaluation modules $V(\ell - 1, a q^{-1})$, $V(1, a q^{\ell-1})$ for \mathcal{L} . Let $V(\ell - 1, a q^{-1}) = \langle u_0, u_1, \dots, u_{\ell-1} \rangle$, $V(1, a q^{\ell-1}) = \langle v_0, v_1 \rangle$ be standard bases of the evaluation modules. By direct calculations, we have the following lemma.

Lemma 5.6 *Consider the tensor product $V(\ell - 1, a q^{-1}) \otimes V(1, a q^{\ell-1})$ of evaluation modules as an \mathcal{L} -module via the coproduct Δ . Set*

$$w_i = q^{-i} u_i \otimes v_0 + u_{i-1} \otimes v_1 \in V(\ell - 1, a q^{-1}) \otimes V(1, a q^{\ell-1})$$

for $0 \leq i \leq \ell$, where $u_{-1} = u_\ell = 0$. Then $\mathbb{C}w_0$ is the highest weight space of the \mathcal{L} -module $V(\ell - 1, a q^{-1}) \otimes V(1, a q^{\ell-1})$. Set $W = \mathcal{L}w_0$. Then

$$W \simeq V(\ell, a)$$

as \mathcal{L} -modules with

$$W = \langle w_0, w_1, \dots, w_\ell \rangle$$

a standard basis for W .

Consider the $\mathcal{L} \otimes \mathcal{L}$ -modules $V \otimes V(\ell, a)$ and $V \otimes (V(\ell - 1, a q^{-1}) \otimes V(1, a q^{\ell-1}))$. Regard them as \mathcal{L} -modules via the coproduct $\Delta : \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{L}$ and then as \mathcal{T} -modules via φ_s . Choose nonzero vectors u, w from the highest weight spaces of V , $V(\ell - 1, a q^{-1}) \otimes V(1, a q^{\ell-1})$ respectively. Then $\mathbb{C}u \otimes w$ is the highest weight space of $V \otimes (V(\ell - 1, a q^{-1}) \otimes V(1, a q^{\ell-1}))$ as an \mathcal{L} -module and hence as a \mathcal{T} -module via φ_s . Set $W = \mathcal{L}w$. The properties (D)₀, (D)₁ hold for the \mathcal{T} -module $V \otimes (V(\ell - 1, a q^{-1}) \otimes V(1, a q^{\ell-1}))$ and its \mathcal{T} -submodule $V \otimes W$. The Drinfel'd polynomials of these \mathcal{T} -modules coincide, since they share the common highest weight space and have the same diameter. On the other hand, $V \otimes V(\ell, a)$ is isomorphic to $V \otimes W$ as \mathcal{L} -modules by Lemma 5.6 and so as \mathcal{T} -modules via φ_s . In particular, the Drinfel'd polynomial of $V \otimes V(\ell, a)$ coincides with that of $V \otimes W$ as \mathcal{T} -modules via φ_s . Therefore $V \otimes V(\ell, a)$ and $V \otimes (V(\ell - 1, a q^{-1}) \otimes V(1, a q^{\ell-1}))$ have the same Drinfel'd polynomial as \mathcal{T} -modules via φ_s . The Drinfel'd polynomial of the \mathcal{T} -module $V \otimes (V(\ell - 1, a q^{-1}) \otimes V(1, a q^{\ell-1}))$ is the product of those of the \mathcal{T} -modules $V \otimes V(\ell - 1, a q^{-1})$ and $V \otimes V(1, a q^{\ell-1})$ by (28), since $V \otimes (V(\ell - 1, a q^{-1}) \otimes V(1, a q^{\ell-1}))$ is isomorphic to $(V \otimes V(\ell - 1, a q^{-1})) \otimes V(1, a q^{\ell-1})$ as \mathcal{T} -modules via φ_s . By induction on ℓ , the formula (29) holds for $\ell - 1$, so we have $P_{V \otimes V(\ell-1, a q^{-1})} = P_V P_{V(\ell-1, a q^{-1})}$. Therefore $P_{V \otimes V(\ell, a)} = P_V P_{V(\ell-1, a q^{-1})} P_{V(1, a q^{\ell-1})}$. On the other hand, $P_{V(\ell-1, a q^{-1})} P_{V(1, a q^{\ell-1})} = P_{V(\ell-1, a q^{-1}) \otimes V(1, a q^{\ell-1})} = P_W = P_{V(\ell, a)}$ by the same argument. This proves the formula (29).

Finally we treat the general case $V' = V'' \otimes V(\ell, a)$, where V'' is afforded by a tensor product of evaluation modules via φ_s . By (29), $P_{V \otimes V'} = P_{V \otimes V'' \otimes V(\ell, a)} = P_{V \otimes V''} P_{V(\ell, a)}$. By induction on $\dim V''$, $P_{V \otimes V''} = P_V P_{V''}$. So $P_{V \otimes V'} = P_V P_{V''} P_{V(\ell, a)}$. By (29), $P_{V''} P_{V(\ell, a)} = P_{V'' \otimes V(\ell, a)} = P_{V'}$. So $P_{V \otimes V'} = P_V P_{V'}$. This completes the proof of Theorem 5.2 (i).

By Lemma 5.6 and Theorem 5.2 (i), we have

$$P_{V(\ell, a)}(\lambda) = \prod_{c \in S(\ell, a)} P_{V(1, c)}(\lambda).$$

By (27), $P_{V(1, c)}(\lambda) = \lambda + c + \varepsilon \varepsilon^* c^{-1}$. This completes the proof of Theorem 5.2 (ii). \square

Proof of the surjectivity of σ in Theorem 1.9. Given an arbitrary monic polynomial $P(\lambda)$ of degree d and an arbitrary nonzero $s \in \mathbb{C}$ such that $P(\varepsilon s^{-2} + \varepsilon^* s^2) \neq 0$, we show that there exists an irreducible \mathcal{T} -module V of type s and diameter d that has Drinfel'd polynomial $P(\lambda)$, i.e., $P_V(\lambda) = P(\lambda)$. Let $\lambda_1, \lambda_2, \dots, \lambda_d$ denote the roots of $P(\lambda)$, allowing repetition. For each i ($1 \leq i \leq d$), choose $a_i \in \mathbb{C}$ such that

$$\lambda_i + a_i + \varepsilon \varepsilon^* a_i^{-1} = 0.$$

If $(\varepsilon, \varepsilon^*) = (1, 1)$, the equation $\lambda_i + a_i + a_i^{-1} = 0$ has nonzero solutions for a_i : we choose one of them and fix it. If $(\varepsilon, \varepsilon^*) = (1, 0)$ or $(0, 0)$, we understand that the equation is $\lambda_i + a_i = 0$ and $a_i = -\lambda_i$. Observe that if $(\varepsilon, \varepsilon^*) = (0, 0)$, then $\lambda_i \neq 0$ ($1 \leq i \leq d$) by the condition $P(\varepsilon s^{-2} + \varepsilon^* s^2) \neq 0$, so $a_i \neq 0$ ($1 \leq i \leq d$). Consider the \mathcal{T} -module V via φ_s , where $\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$ and

$$V = V(1, a_1) \otimes V(1, a_2) \otimes \dots \otimes V(1, a_d).$$

By Theorem 5.2, it holds that $P_V(\lambda) = P(\lambda)$. Choose a nonzero vector w from the highest weight space of V and set $W = \mathcal{T}w$. Let M be a maximal \mathcal{T} -submodule of W . Observe $w \notin M$, since $M \neq W$. The quotient \mathcal{T} -module $\overline{W} = W/M$ is irreducible. Since $P_V(\varepsilon s^{-2} + \varepsilon^* s^2) \neq 0$, we have $\sigma_d(V) \neq 0$ by Remark 1.10. By Lemma 5.1, $P_V(\lambda) = P_{\overline{W}}(\lambda)$. Thus \overline{W} is the desired \mathcal{T} -module. \square

6 Irreducibility of a \mathcal{T} -module $\tilde{V} = V \otimes V(\ell, a)$ via φ_s

For the augmented TD-algebra $\mathcal{T} = \mathcal{T}_q^{\varepsilon, \varepsilon^*}$, we have so far established the bijectivity of the mapping

$$\text{Irr}_d^s(\mathcal{T}) \longmapsto \mathcal{P}_d^s \quad (V \longmapsto P_V(\lambda)),$$

namely, the set of finite-dimensional irreducible \mathcal{T} -modules of type s and diameter d is parametrized up to isomorphism by monic polynomials of degree d that do not vanish at $\varepsilon s^{-2} + \varepsilon^* s^2$. Given a polynomial $P(\lambda) \in \mathcal{P}_d^s$, we want to construct explicitly a \mathcal{T} -module via φ_s that belongs to $\text{Irr}_d^s(\mathcal{T})$ and has $P(\lambda)$ as its Drinfel'd polynomial. In this section, we prepare a key proposition to the construction of such \mathcal{T} -modules via φ_s . The construction itself will be discussed in the next section. We consider \mathcal{T} -modules

$$\begin{aligned} V &= V(\ell_1, a_1) \otimes \dots \otimes V(\ell_n, a_n), \\ \tilde{V} &= V \otimes V(1, a) \end{aligned}$$

via φ_s , where $1 \leq n$, $1 \leq \ell_i$ ($1 \leq i \leq n$) and a, a_i are allowed to be zero if $(\varepsilon, \varepsilon^*) = (1, 0)$. The diameters of V, \tilde{V} are $d = \ell_1 + \cdots + \ell_n$, $d + 1$ respectively. Observe that $\sigma_{d+1}(\tilde{V}) = \tilde{Q} P_{\tilde{V}}(\varepsilon s^{-2} + \varepsilon^* s^2)$ for some nonzero scalar \tilde{Q} by Remark 1.10. We have $P_{\tilde{V}}(\lambda) = P_V(\lambda) P_{V(1,a)}(\lambda)$ by Theorem 5.2. So again by Remark 1.10, $\sigma_{d+1}(\tilde{V}) \neq 0$ if and only if $\sigma_d(V) \neq 0$ and $\sigma_1(V(1,a)) \neq 0$. By Theorem 1.9, observe also that $\sigma_d(V) \neq 0$ holds if the \mathcal{T} -module V is irreducible.

Proposition 6.1 *Assume that a \mathcal{T} -module V via φ_s is irreducible and has diameter d . Assume also that the \mathcal{T} -module $\tilde{V} = V \otimes V(1,a)$ via φ_s satisfies $\sigma_{d+1}(\tilde{V}) \neq 0$. If the \mathcal{T} -module \tilde{V} via φ_s has a nonzero \mathcal{T} -submodule W that does not contain the highest weight space \tilde{U}_0 of \tilde{V} , then the Drinfel'd polynomial $P_V(\lambda)$ of the \mathcal{T} -module V via φ_s vanishes at $\lambda = -a q^2 - \varepsilon \varepsilon^* a^{-1} q^{-2}$:*

$$P_V(-a q^2 - \varepsilon \varepsilon^* a^{-1} q^{-2}) = 0,$$

where we understand $\varepsilon^* a^{-1} = 0$ if $(\varepsilon, \varepsilon^*) = (1, 0)$ and $a = 0$.

The remainder of this section is devoted to the proof of Proposition 6.1. Without loss of generality, we can assume that W is irreducible as a \mathcal{T} -submodule, since we may replace W by a minimal \mathcal{T} -submodule contained in W . Let

$$\begin{aligned} \tilde{V} &= \bigoplus_{i=0}^{d+1} \tilde{U}_i, \\ W &= \bigoplus_{i=r}^{r+d'} U_i(W) \quad (U_i(W) \subseteq \tilde{U}_i) \end{aligned}$$

denote the weight-space decompositions of \tilde{V}, W respectively, where d' is the diameter of W . Since W is irreducible, the highest weight space $U_r(W)$ has dimension 1 by Theorem 1.8 and so is spanned by a nonzero vector w_0 :

$$U_r(W) = \langle w_0 \rangle.$$

Since $W \not\subseteq \tilde{U}_0$, we have $r \geq 1$. Since $x U_i(W) \subseteq U_{i+1}(W)$ and $y U_i(W) \subseteq U_{i-1}(W)$, where $u_{r-1}(W) = 0$, $U_{r+d'+1}(W) = 0$, we have

$$y w_0 = 0, \tag{30}$$

$$y^i x^i w_0 = \sigma_i(W) w_0 \tag{31}$$

for $i = 0, 1, 2, \dots$. Let

$$V = \bigoplus_{i=0}^d U_i$$

denote the weight space decomposition of V and let

$$V(1, a) = \langle v_0, v_1 \rangle$$

be a standard basis of $V(1, a)$. Then

$$\tilde{U}_i = U_i \otimes \langle v_0 \rangle + U_{i-1} \otimes \langle v_1 \rangle$$

for $0 \leq i \leq d+1$, where $U_{-1} = U_{d+1} = 0$. In particular

$$w_0 = u_r \otimes v_0 + u_{r-1} \otimes v_1 \quad (32)$$

for some $u_r \in U_r$, $u_{r-1} \in U_{r-1}$.

Lemma 6.2 For $i, m \in \mathbb{Z}$, set

$$\begin{aligned} c_i(m) &= a s q^{i+2m-d-1} + \varepsilon s^{-1} q^{-i-2m+d+1}, \\ c_i^*(m) &= \varepsilon^* a^{-1} s q^{-i+2m-d+1} + s^{-1} q^{i-2m+d-1}. \end{aligned}$$

Then for $1 \leq i$, the following (i) \sim (v) hold.

- (i) $y u_{r-1} = 0$,
 $y u_r = -c_1^*(r-1) u_{r-1}$.
- (ii) $\sigma_i(W) u_{r-1} = y^i x^i u_{r-1} + \alpha q [i] c_i(r) y^i x^{i-1} u_r$.
- (iii) $\sigma_i(W) u_r = y^i x^i u_r + \alpha q [i]^2 c_i(r) c_{-i}^*(r-1) y^{i-1} x^{i-1} u_r + [i] c_{-i}^*(r-1) y^{i-1} x^i u_{r-1}$.
- (iv) $y^{i+1} x^i u_r = -[i+1] c_{-(i-1)}^*(r-1) \sigma_i(W) u_{r-1}$.
- (v) $y^i x^i u_{r-1} = \sigma_i(W) u_{r-1} + \alpha q [i]^2 c_i(r) c_{-(i-2)}^*(r-1) \sigma_{i-1}(W) u_{r-1}$.

Proof. By Lemma 5.4, we have for $w_0 = u_r \otimes v_0 + u_{r-1} \otimes v_1$,

$$y w_0 = (y u_r) \otimes v_0 + (y u_{r-1}) \otimes v_1 + c_1^*(r-1) u_{r-1} \otimes v_0.$$

Since $y w_0 = 0$ and $y u_r \in U_{r-1}$, we obtain $y u_r + c_1^*(r-1) u_{r-1} = 0$, $y u_{r-1} = 0$ and (i) holds. Again by Lemma 5.4, we have

$$\begin{aligned} x^i w_0 &= (x^i u_r) \otimes v_0 + \alpha q [i] c_i(r) (x^{i-1} u_r) \otimes v_1 + (x^i u_{r-1}) \otimes v_1, \\ y^i x^i w_0 &= (y^i x^i u_r) \otimes v_0 \\ &\quad + \alpha q [i] c_i(r) ((y^i x^{i-1} u_r) \otimes v_1 + [i] c_i^*(r+i-1) (y^{i-1} x^{i-1} u_r) \otimes v_0) \\ &\quad + (y^i x^i u_{r-1}) \otimes v_1 + [i] c_i^*(r+i-1) (y^{i-1} x^i u_{r-1}) \otimes v_0. \end{aligned}$$

Since $y^i x^i w_0 = \sigma_i(W) w_0$ and $y^i x^i u_r, y^{i-1} x^{i-1} u_r, y^{i-1} x^i u_{r-1} \in U_r$, $y^i x^{i-1} u_r, y^i x^i u_{r-1} \in U_{r-1}$, we obtain

$$\begin{aligned} \sigma_i(W) u_r &= y^i x^i u_r + \alpha q [i]^2 c_i(r) c_i^*(r+i-1) y^{i-1} x^{i-1} u_r \\ &\quad + [i] c_i^*(r+i-1) y^{i-1} x^i u_{r-1}, \\ \sigma_i(W) u_{r-1} &= \alpha q [i] c_i(r) y^i x^{i-1} u_r + y^i x^i u_{r-1}. \end{aligned}$$

Since $c_i^*(r+i-1) = c_{-i}^*(r-1)$, (ii) and (iii) hold.

By (ii) and (iii) we obtain

$$\sigma_i(W) y u_r - [i] c_{-i}^*(r-1) \sigma_i(W) u_{r-1} = y^{i+1} x^i u_r.$$

Since $y u_r = -c_1^*(r-1) u_{r-1}$ by (i) and $c_1^*(r-1) + [i] c_{-i}^*(r-1) = [i+1] c_{-(i-1)}^*(r-1)$, we have (iv). Observe $y^i x^{i-1} u_r = -[i] c_{-(i-2)}^*(r-1) \sigma_{i-1}(W) u_{r-1}$ is valid for $i \geq 1$ by (iv), (i) and put this identity into (ii) to obtain (v). \square

Lemma 6.3 *It holds that (i) $u_{r-1} \neq 0$, (ii) $u_r \neq 0$ and (iii) $r = 1$.*

Proof. Suppose $u_{r-1} = 0$. Then by Lemma 6.2 (iii),

$$y^i x^i u_r = -\alpha q [i]^2 c_i(r) c_{-i}^*(r-1) y^{i-1} x^{i-1} u_r + \sigma_i(W) u_r$$

for $1 \leq i$, so we have $y^i x^i u_r \in \langle u_r \rangle$ for $0 \leq i$ by induction on i . Moreover $y u_r = 0$ by Lemma 6.2 (i). Since \mathcal{T} is spanned by $k^n \omega_\lambda(x, y)$ ($n \in \mathbb{Z}$, $\lambda \in \Lambda^{irr}$) by Theorem 2.2, it follows from $y^i x^i u_r \in \langle u_r \rangle$ ($0 \leq i$) and $y u_r = 0$ that

$$\mathcal{T} u_r \subseteq \bigoplus_{r \leq i} U_i.$$

Since $1 \leq r$, $\mathcal{T} u_r$ is a proper \mathcal{T} -submodule of V . This contradicts the irreducibility of V . Thus (i) holds.

Suppose $u_r = 0$. Then by Lemma 6.2 (ii),

$$y^i x^i u_{r-1} = \sigma_i(W) u_{r-1}.$$

Since \mathcal{T} is spanned by $k^n \omega_\lambda(x, y)$ ($n \in \mathbb{Z}$, $\lambda \in \Lambda^{irr}$), it follows from $y^i x^i u_{r-1} \in \langle u_{r-1} \rangle$ ($0 \leq i$) and $y u_{r-1} = 0$ that

$$\mathcal{T} u_{r-1} \subseteq \bigoplus_{r-1 \leq i} U_i.$$

So V has a nonzero \mathcal{T} -submodule contained in $\bigoplus_{r-1 \leq i} U_i$. Since V is irreducible, we obtain $r-1 = 0$, i.e., $w_0 = u_0 \otimes v_1$ ($u_0 \neq 0, u_1 = 0$). By Lemma 6.2 (i), $y u_1 = -c_1^*(0) u_0$ and so $c_1^*(0) = 0$. By Lemma 6.2 (iii) with $i = 1$, $c_{-1}^*(0) x u_0 = 0$. Note $x u_0 \neq 0$, otherwise $\sigma_d(V) = 0$, which contradicts the assumption that V is irreducible as a \mathcal{T} -module. Thus $c_{-1}^*(0) = 0$. From $c_1^*(0) = 0$ and $c_{-1}^*(0) = 0$, we have

$$\begin{aligned} \varepsilon^* a^{-1} s q^{-d} + s^{-1} q^d &= 0, \\ \varepsilon^* a^{-1} s q^{-d+2} + s^{-1} q^{d-2} &= 0. \end{aligned}$$

This implies $\varepsilon^* = 1$, $a = -s^2 q^{-2d} = -s^2 q^{4-2d}$ and we have $q^4 = 1$. This contradicts the assumption that q is not a root of unity. Hence (ii) holds.

By Lemma 6.2 (i), (v), we have $y u_{r-1} = 0$ and $y^i x^i u_{r-1} \in \langle u_{r-1} \rangle$ ($0 \leq i$). The same argument of the previous paragraph is valid and V has a nonzero \mathcal{T} -submodule $\mathcal{T} u_{r-1}$ contained in $\bigoplus_{r-1 \leq i} U_i$. Hence we obtain $r-1 = 0$, i.e., (iii) holds. \square

Lemma 6.4 For $0 \leq i$,

$$\sigma_i(W) = f_i \sigma_{i-1}(W) + \sigma_i(V),$$

where

$$f_i = (q^i - q^{-i})^2 (\varepsilon s^{-2} q^{2(d-i)} + \varepsilon^* s^2 q^{-2(d-i)} + a q^2 + \varepsilon \varepsilon^* a^{-1} q^{-2}).$$

Proof. Since $r = 1$, we have $y^i x^i u_{r-1} = \sigma_i(V) u_{r-1}$. By Lemma 6.3, $u_{r-1} \neq 0$. By Lemma 6.2 (v), we obtain

$$\begin{aligned} \sigma_i(V) &= \sigma_i(W) + \alpha q [i]^2 c_i(1) c_{-(i-2)}^*(0) \sigma_{i-1}(W) \\ &= \sigma_i(W) - f_i \sigma_{i-1}(W). \end{aligned}$$

□

Lemma 6.5 It holds that $d' = d - 1$, where d, d' are the diameters of V, W respectively.

Proof. Obviously $\sigma_i(W) = \sigma_{i-1}(W) = 0$ for $d' + 2 \leq i$. So we have $\sigma_i(V) = 0$ for $d' + 2 \leq i$ by Lemma 6.4. This implies $d \leq d' + 1$, since $\sigma_d(V) \neq 0$. On the other hand, the weight-space decompositions of \tilde{V}, W are $\tilde{V} = \tilde{U}_0 + \cdots + \tilde{U}_{d+1}$, $W = U_r(W) + \cdots + U_{r+d'}(W)$ ($r = 1$) with $U_i(W) \subseteq \tilde{U}_i$. So $r + d' \leq d + 1$, i.e., $d' \leq d$. Therefore either $d = d' + 1$ or $d = d'$.

Suppose $d = d'$. Then $0 \neq U_{d+1}(W) \subseteq \tilde{U}_{d+1}$. Since \tilde{V} is a tensor product of evaluation modules, we generally have $\dim \tilde{U}_0 = \dim \tilde{U}_{d+1} = 1$. So $U_{d+1}(W) = \tilde{U}_{d+1}$, in particular $\mathcal{T}\tilde{U}_{d+1}$ is contained in W . On the other hand, we assumed $\sigma_{d+1}(\tilde{V}) \neq 0$ for Proposition 6.1. This implies $\mathcal{T}\tilde{U}_{d+1} \supseteq \tilde{U}_0$. Therefore W contains \tilde{U}_0 , which is a contradiction. □

Proof of Proposition 6.1. Set $\sigma_i = \sigma_i(V)$. Using Lemma 6.4 repeatedly, we have for $0 \leq i$

$$\sigma_i(W) = (f_i f_{i-1} \cdots f_1) \sigma_0 + (f_i f_{i-1} \cdots f_2) \sigma_1 + \cdots + f_i \sigma_{i-1} + \sigma_i.$$

By Lemma 6.5, $d' = d - 1$. So $\sigma_d(W) = 0$. Thus

$$\sum_{j=0}^d f_d f_{d-1} \cdots f_{j+1} \sigma_j = 0. \quad (33)$$

Define the polynomial $f_i(\lambda)$ of degree 1 in λ by

$$f_i(\lambda) = (q^i - q^{-i})^2 (\varepsilon s^{-2} q^{2(d-i)} + \varepsilon^* s^2 q^{-2(d-i)} - \lambda)$$

for $1 \leq i$. Then by definition

$$P_V(\lambda) = Q^{-1} \sum_{i=0}^d \sigma_i f_{i+1}(\lambda) \cdots f_d(\lambda),$$

where $Q = (-1)^d (q - q^{-1})^2 (q^2 - q^{-2})^2 \cdots (q^d - q^{-d})^2$. Since $f_i = f_i(-a q^2 - \varepsilon \varepsilon^* a^{-1} q^{-2})$, we have by (33)

$$P_V(-a q^2 - \varepsilon \varepsilon^* a^{-1} q^{-2}) = 0.$$

This completes the proof of Proposition 6.1. □

7 Construction of finite-dimensional irreducible \mathcal{T} -modules via φ_s

In this section, we prove Theorem 1.15, Theorem 1.18, Theorem 1.21. Each theorem consists of three parts (i), (ii), (iii). The second part (ii) immediately follows from the first part (i) together with Theorem 1.9' and Lemma 1.14. We only prove (i) and (iii) for each of the theorems. The Drinfel'd polynomials $P_V(\lambda)$ in the first part (i) are obtained straightforward by Theorem 5.2. Throughout this section, s stands for a nonzero scalar of \mathbb{C} chosen arbitrarily.

For the augmented TD-algebra $\mathcal{T} = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$, we consider the following \mathcal{T} -module V via φ_s (see Section 1.4): if $(\varepsilon, \varepsilon^*) = (1, 1)$ or $(0, 0)$,

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n),$$

where $1 \leq n$, $1 \leq \ell_i$, $a_i \neq 0$ ($1 \leq i \leq n$), and if $(\varepsilon, \varepsilon^*) = (1, 0)$,

$$V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n),$$

where $0 \leq n$, $0 \leq \ell$, $1 \leq \ell_i$, $a_i \neq 0$ ($1 \leq i \leq n$). With such a \mathcal{T} -module V via φ_s , we associate the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings, where

$$S(\ell_i, a_i) = \{a_i q^{-\ell_i+1}, a_i q^{-\ell_i+3}, \dots, a_i q^{\ell_i-1}\}.$$

Consider a \mathcal{T} -module V' via φ_s of the same kind:

$$\begin{aligned} V' &= V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_m, a'_m) & \text{if } (\varepsilon, \varepsilon^*) = (1, 1), (0, 0), \\ V' &= V(\ell') \otimes V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_m, a'_m) & \text{if } (\varepsilon, \varepsilon^*) = (1, 0). \end{aligned}$$

For $(\varepsilon, \varepsilon^*) = (1, 1)$, such \mathcal{T} -modules V, V' via φ_s are said to be *equivalent* if the associated multi-sets of q -strings are equivalent, i.e., $m = n$ and there exist $\varepsilon_i \in \{1, -1\}$ ($1 \leq i \leq n$) such that $S(\ell'_i, a'_i) = S(\ell_i, a_i^{\varepsilon_i})$ ($1 \leq i \leq n$) with a suitable rearrangement of the ordering of $S(\ell'_1, a'_1), \dots, S(\ell'_n, a'_n)$. For $(\varepsilon, \varepsilon^*) = (0, 0)$, such \mathcal{T} -modules V, V' via φ_s are said to be *equivalent* if $m = n$ and $S(\ell'_i, a'_i) = S(\ell_i, a_i)$ ($1 \leq i \leq n$) with a suitable rearrangement of the ordering of $S(\ell'_1, a'_1), \dots, S(\ell'_n, a'_n)$. For $(\varepsilon, \varepsilon^*) = (1, 0)$, such \mathcal{T} -modules V, V' via φ_s are said to be *equivalent* if $\ell = \ell'$, $m = n$ and $S(\ell'_i, a'_i) = S(\ell_i, a_i)$ ($1 \leq i \leq n$) with a suitable rearrangement of the ordering of $S(\ell'_1, a'_1), \dots, S(\ell'_n, a'_n)$.

Lemma 7.1 *If a \mathcal{T} -module V via φ_s is irreducible, then every \mathcal{T} -module V' via φ_s that is equivalent to V is isomorphic to V as \mathcal{T} -modules via φ_s , in particular irreducible.*

Proof. Since V and V' are equivalent, V and V' have the same Drinfel'd polynomial by Theorem 5.2. In particular $\sigma_d(V) = \sigma_d(V')$, where d is the diameter of the \mathcal{T} -modules V, V' . Let U'_0 denote the highest weight space of the \mathcal{T} -module V' via φ_s . Set $W = \mathcal{T}U'_0$ and let M be a maximal \mathcal{T} -submodule of W . Then V' and W/M have the same Drinfel'd polynomial by Lemma 5.1. Hence V and W/M have the same Drinfel'd polynomial. By Theorem 1.9', the irreducible \mathcal{T} -modules $V, W/M$ are isomorphic, in particular $\dim V = \dim W/M$. But V and V' are equivalent, in particular $\dim V = \dim V'$. Thus $\dim V' = \dim W/M$ and we have $V' = W, M = 0$. This means that V and V' are isomorphic as \mathcal{T} -modules via φ_s . \square

Proof of the ‘only if’ part of (i). The ‘only if’ part of Theorem 1.15 (i) follows from Lemma 7.1. Suppose $-s^2 \in S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})$ for some i ($1 \leq i \leq n$). Then $P_V(\varepsilon s^{-2} + \varepsilon^* s^2) = 0$ by Theorem 5.2, which contradicts the irreducibility of V by Theorem 1.9'. Suppose the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is not strongly in general position. Then there exists a multi-set $\{S(\ell'_i, a'_i)\}_{i=1}^m$ of q -strings that is equivalent to $\{S(\ell_i, a_i)\}_{i=1}^n$ and not in general position. Set $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_m, a'_m)$. Since $\{S(\ell'_i, a'_i)\}_{i=1}^m$ is not in general position, V' is not irreducible as an \mathcal{L} -module, consequently as a \mathcal{T} -module via φ_s . Since $\{S(\ell'_i, a'_i)\}_{i=1}^m$ is equivalent to $\{S(\ell_i, a_i)\}_{i=1}^n$ and V is irreducible as a \mathcal{T} -module via φ_s , V' is also irreducible as a \mathcal{T} -module via φ_s by Lemma 7.1. Thus we get a contradiction. The ‘only if’ part of Theorem 1.18 (i), Theorem 1.21 (i) can be proved similarly. \square

7.1 Proof of the ‘if’ part of Theorem 1.15 (i)

We start with an observation of the $U_q(sl_2)$ -loop algebra \mathcal{L} .

Lemma 7.2 *There exists an algebra anti-homomorphism of \mathcal{L} that sends e_i^+ , $e_i^- k_i$, k_i , k_i^{-1} to $e_i^- k_i$, e_i^+ , k_i , k_i^{-1} , respectively ($i = 0, 1$). Such an anti-homomorphism is unique and we denote it by τ :*

$$\tau : \mathcal{L} \longrightarrow \mathcal{L} \quad (e_i^+, e_i^- k_i, k_i^{\pm 1} \mapsto e_i^- k_i, e_i^+, k_i^{\pm 1} \text{ respectively}).$$

It holds that $\tau^2 = 1$ and $(\tau \otimes \tau) \Delta = \Delta \tau$, where $\Delta : \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{L}$ is the coproduct of \mathcal{L} .

The assertions of Lemma 7.2 can be checked by straightforward calculations.

For an \mathcal{L} -module V , the dual vector space of V

$$\text{Hom}(V, \mathbb{C}) = \{f : V \longrightarrow \mathbb{C} \mid f \text{ is a linear mapping}\}$$

becomes an \mathcal{L} -module by

$$(Xf)(v) = f(\tau(X)v) \quad (v \in V)$$

for $f \in \text{Hom}(V, \mathbb{C})$, $X \in \mathcal{L}$. For \mathcal{L} -modules V , V' , we identify $\text{Hom}(V \otimes V', \mathbb{C})$ with $\text{Hom}(V, \mathbb{C}) \otimes \text{Hom}(V', \mathbb{C})$ as vector spaces by

$$(f \otimes g)(v \otimes v') = f(v)g(v').$$

It can be easily checked by the relation $(\tau \otimes \tau) \Delta = \Delta \tau$ that this identification gives an \mathcal{L} -module isomorphism

$$\text{Hom}(V \otimes V', \mathbb{C}) \simeq \text{Hom}(V, \mathbb{C}) \otimes \text{Hom}(V', \mathbb{C}),$$

where \mathcal{L} acts on $V \otimes V'$ and $\text{Hom}(V, \mathbb{C}) \otimes \text{Hom}(V', \mathbb{C})$ via the coproduct Δ .

Lemma 7.3 *For evaluation modules, we have the following isomorphisms as \mathcal{L} -modules.*

- (i) $\text{Hom}(V(\ell, a), \mathbb{C}) \simeq V(\ell, a^{-1})$.
- (ii) $\text{Hom}(V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n), \mathbb{C}) \simeq V(\ell_1, a_1^{-1}) \otimes \cdots \otimes V(\ell_n, a_n^{-1})$.

Proof. Let $V(\ell, a) = \langle v_0, v_1, \dots, v_\ell \rangle$ be a standard basis and $\text{Hom}(V(\ell, a), \mathbb{C}) = \langle f_0, f_1, \dots, f_\ell \rangle$ the dual basis: $f_i(v_j) = \delta_{ij}$. Set

$$g_i = q^{-i(\ell-i+1)} \binom{\ell}{i} f_i,$$

where $\binom{\ell}{i} = [\ell]!/[i]![\ell-i]!$, the q -binomial coefficient. Then we have $e_0^+ g_i = a^{-1} q [i+1] g_{i+1}$, $e_0^- g_i = a q^{-1} [\ell-i+1] g_{i-1}$, $e_1^+ g_i = [\ell-i+1] g_{i-1}$, $e_1^- g_i = [i+1] g_{i+1}$, $k_0 g_i = q^{2i-\ell} g_i$, where $g_{-1} = g_{\ell+1} = 0$. So if $V(\ell, a^{-1}) = \langle w_0, w_1, \dots, w_\ell \rangle$ is a standard basis, then $\text{Hom}(V(\ell, a), \mathbb{C})$ is isomorphic to $V(\ell, a^{-1})$ as \mathcal{L} -modules by the correspondence of g_i to w_i ($0 \leq i \leq \ell$). Part (ii) follows from part (i) and the fact $\text{Hom}(V \otimes V', \mathbb{C}) \simeq \text{Hom}(V, \mathbb{C}) \otimes \text{Hom}(V', \mathbb{C})$ as \mathcal{L} -modules. \square

We now prove the ‘if’ part of Theorem 1.15 (i). Namely, we are in the case of $(\varepsilon, \varepsilon^*) = (1, 1)$ and given a \mathcal{T} -module

$$V = V(\ell_1, a_1) \otimes \dots \otimes V(\ell_n, a_n)$$

via φ_s such that $-s^2 \notin S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})$ ($1 \leq i \leq n$) and the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is strongly in general position. We want to show that V is irreducible as a \mathcal{T} -module via φ_s . Observe that the ordering of the tensor product does not change the isomorphism class of V as an \mathcal{L} -module and consequently as a \mathcal{T} -module via φ_s , since the multiset $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is in general position. First we show

Lemma 7.4 *For any choice of $\varepsilon_i \in \{1, -1\}$ ($1 \leq i \leq n$), V is isomorphic to*

$$V(\ell_1, a_1^{\varepsilon_1}) \otimes \dots \otimes V(\ell_n, a_n^{\varepsilon_n})$$

as \mathcal{T} -modules via φ_s .

Proof. We proceed by induction on n . First let $n = 1$. Then $V(\ell_1, a_1)$, $V(\ell_1, a_1^{-1})$ have the same Drinfel’d polynomial by Theorem 5.2 and the Drinfel’d polynomial does not vanish at $s^2 + s^{-2}$, since $-s^2 \notin S(\ell_1, a_1) \cup S(\ell_1, a_1^{-1})$. So $V(\ell_1, a_1)$, $V(\ell_1, a_1^{-1})$ have nonzero σ_{ℓ_1} by Remark 1.10. In the case of evaluation modules, the property $\sigma_{\ell_1} \neq 0$ implies the irreducibility of the \mathcal{T} -modules via φ_s . Thus $V(\ell_1, a_1)$, $V(\ell_1, a_1^{-1})$ are irreducible as \mathcal{T} -modules via φ_s and have the same Drinfel’d polynomial that does not vanish at $s^2 + s^{-2}$. By Theorem 1.9’, $V(\ell_1, a_1)$, $V(\ell_1, a_1^{-1})$ are isomorphic as \mathcal{T} -modules via φ_s .

For $n \geq 2$, set $V' = V(\ell_1, a_1) \otimes \dots \otimes V(\ell_{n-1}, a_{n-1})$. Then by induction on n , V' is isomorphic to $V'' = V(\ell_1, a_1^{\varepsilon_1}) \otimes \dots \otimes V(\ell_{n-1}, a_{n-1}^{\varepsilon_{n-1}})$ as \mathcal{T} -modules via φ_s . Let $\psi : V' \longrightarrow V''$ denote an isomorphism between the \mathcal{T} -modules V' , V'' via φ_s . The generators x, y, k, k^{-1} of \mathcal{T} are mapped by φ_s to $x(s) = \alpha(s e_0^+ + s^{-1} e_1^- k_1)$, $y(s) = s e_0^- k_0 + s^{-1} e_1^+ k_1$, sk_0 , $s^{-1} k_1$, respectively, and those elements of \mathcal{L} are mapped by Δ to

$$\begin{aligned} \Delta(x(s)) &= x(s) \otimes 1 + \alpha s k_0 \otimes e_0^+ + \alpha s^{-1} k_1 \otimes e_1^- k_1, \\ \Delta(y(s)) &= y(s) \otimes 1 + s k_0 \otimes e_0^- k_0 + s^{-1} k_1 \otimes e_1^+, \\ \Delta(s k_0) &= s k_0 \otimes k_0, \\ \Delta(s^{-1} k_1) &= s^{-1} k_1 \otimes k_1, \end{aligned}$$

respectively. It can be easily checked that the vector-space isomorphism

$$\psi \otimes id : V' \otimes V(\ell_n, a_n) \longrightarrow V'' \otimes V(\ell_n, a_n)$$

commutes with the action of each of the elements $\Delta(x(s)), \Delta(y(s)), \Delta(sk_0), \Delta(s^{-1}k_1)$. So we get

$$V' \otimes V(\ell_n, a_n) \simeq V'' \otimes V(\ell_n, a_n)$$

as \mathcal{T} -modules via φ_s . Since $\{S(\ell_i, a_i^{\varepsilon_i})\}_{i=1}^{n-1} \cup \{S(\ell_n, a_n)\}$ is in general position,

$$V'' \otimes V(\ell_n, a_n) \simeq V(\ell_n, a_n) \otimes V''$$

as \mathcal{L} -modules and consequently as \mathcal{T} -modules via φ_s . By the same argument, we have

$$\begin{aligned} V(\ell_n, a_n) \otimes V'' &\simeq V(\ell_n, a_n^{\varepsilon_n}) \otimes V'' \\ &\simeq V'' \otimes V(\ell_n, a_n^{\varepsilon_n}) \end{aligned}$$

as \mathcal{T} -modules via φ_s . Thus $V' \otimes V(\ell_n, a_n) \simeq V'' \otimes V(\ell_n, a_n^{\varepsilon_n})$ as \mathcal{T} -modules via φ_s and the proof is completed. \square

Next we introduce a partial ordering on $\mathbb{C} \setminus \{0\}$ by

$$a \leq b \Leftrightarrow b = a q^{2i} \text{ for some integer } i \geq 0. \quad (34)$$

Consider i_0 ($1 \leq i_0 \leq n$) such that $a_{i_0} q^{\ell_{i_0}-1}$ or $a_{i_0}^{-1} q^{\ell_{i_0}-1}$ is maximal with respect to the partial ordering on the set of nonzero scalars $a_i q^{\ell_i-1}, a_i^{-1} q^{\ell_i-1}$ ($1 \leq i \leq n$). Among such i_0 's, choose one for which ℓ_{i_0} is smallest. Since the ordering of the tensor product does not matter about the isomorphism class of V as a \mathcal{T} -module via φ_s , we may assume that $i_0 = n$, and by Lemma 7.4 that $a_n q^{\ell_n-1}$ is maximal among $a_i^{\pm} q^{\ell_i-1}$ ($1 \leq i \leq n$). So

$$a_n q^{\ell_n+1} \notin S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1}) \quad (1 \leq i \leq n), \quad (35)$$

$$\ell_n \leq \ell_i \text{ if } a_n q^{\ell_n-1} \in S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1}). \quad (36)$$

We proceed by induction on $\dim V$ to prove the ‘if’ part of Theorem 1.15 (i). Set

$$V' = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_{n-1}, a_{n-1}).$$

Then $V = V' \otimes V(\ell_n, a_n)$. Since the \mathcal{L} -module $V(\ell_n, a_n)$ is by Lemma 5.6 embedded in the \mathcal{L} -module $V(\ell_n - 1, a_n q^{-1}) \otimes V(1, a_n q^{\ell_n-1})$ as the \mathcal{L} -submodule spanned by the highest weight space, the \mathcal{L} -module V can be regarded as embedded in the \mathcal{L} -module

$$\tilde{V} = (V' \otimes V(\ell_n - 1, a_n q^{-1})) \otimes V(1, a_n q^{\ell_n-1}).$$

We understand $V' \otimes V(\ell_n - 1, a_n q^{-1}) = V'$ if $\ell_n = 1$. Our strategy is to apply Proposition 6.1 to the \mathcal{T} -module \tilde{V} via φ_s . To do so, we need to check the prerequisites for it, namely that $V' \otimes V(\ell_n - 1, a_n q^{-1})$ is irreducible as a \mathcal{T} -module via φ_s , and that $\sigma_d(\tilde{V}) \neq 0$ holds, where $d = \ell_1 + \cdots + \ell_n$, the diameter of the \mathcal{T} -module \tilde{V} via φ_s . To show that $V' \otimes V(\ell_n - 1, a_n q^{-1})$ is irreducible as a \mathcal{T} -module via φ_s , it is enough to check the induction hypotheses for it, i.e.,

the first induction hypothesis that $-s^2$ is contained neither in $S(\ell_i, a_i^{\pm 1})$ ($1 \leq i \leq n-1$) nor in $S(\ell_n - 1, (a_n q^{-1})^{\pm 1})$, and the second induction hypothesis that the multi-set $\{S(\ell_i, a_i)\}_{i=1}^{n-1} \cup \{S(\ell_n - 1, a_n q^{-1})\}$ of q -strings associated with $V' \otimes V(\ell_n - 1, a_n q^{-1})$ is strongly in general position. The first induction hypothesis is satisfied, since $-s^2 \notin S(\ell_i, a_i^{\pm 1})$ ($1 \leq i \leq n$) was assumed at the beginning and it generally holds that $S(\ell_n - 1, a_n q^{-1}) = S(\ell_n, a_n) \setminus \{a_n q^{\ell_n - 1}\}$. The second induction hypothesis is satisfied, since the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ was assumed at the beginning to be strongly in general position and n was chosen to satisfy (35), (36). Thus by induction on dimension, $V' \otimes V(\ell_n - 1, a_n q^{-1})$ is irreducible as a \mathcal{T} -module via φ_s . To show $\sigma_d(\tilde{V}) \neq 0$, it is enough to check that $P_{\tilde{V}}(\lambda)$ does not vanish at $\lambda = s^{-2} + s^2$ (see Remark 1.10). Since $-s^2 \notin S(\ell_i, a_i^{\pm 1})$ ($1 \leq i \leq n$), $P_{\tilde{V}}(s^{-2} + s^2) \neq 0$ by Theorem 5.2. We are now ready to apply Proposition 6.1 to \tilde{V} .

Suppose that the \mathcal{T} -module V via φ_s has a nonzero \mathcal{T} -submodule W that does not contain the highest weight space of V . Embed V into the \mathcal{T} -module \tilde{V} via φ_s as a \mathcal{T} -submodule in such a way that V and \tilde{V} share the highest weight space in common (see the last paragraph). Then by Proposition 6.1, the Drinfel'd polynomial of $V' \otimes V(\ell_n - 1, a_n q^{-1})$ vanishes at $-a_n q^{\ell_n + 1} - a_n^{-1} q^{-\ell_n - 1}$. This contradicts (35) by Theorem 5.2. Therefore we conclude that every nonzero \mathcal{T} -submodule W of the \mathcal{T} -module V via φ_s contains the highest weight space of V .

Finally consider the \mathcal{L} -module $\text{Hom}(V, \mathbb{C})$. By Lemma 7.3, $\text{Hom}(V, \mathbb{C})$ is isomorphic to $V(\ell_1, a_1^{-1}) \otimes \cdots \otimes V(\ell_n, a_n^{-1})$ as \mathcal{L} -modules. So $\text{Hom}(V, \mathbb{C})$ and V are isomorphic as \mathcal{T} -modules via φ_s by Lemma 7.4. For a subspace W of V , define the subspace W^\perp of $\text{Hom}(V, \mathbb{C})$ by

$$W^\perp = \{f \in \text{Hom}(V, \mathbb{C}) \mid f(w) = 0 \ (w \in W)\}.$$

If W is invariant by the action of \mathcal{T} via φ_s , then so is W^\perp , because the action of $X \in \mathcal{L}$ on $\text{Hom}(V, \mathbb{C})$ is defined by $(Xf)(v) = f(\tau(X)v)$ ($f \in \text{Hom}(V, \mathbb{C})$, $v \in V$) and $\tau(\mathcal{T}) = \mathcal{T}$ holds by $\tau(x(s)) = \alpha y(s)$, $\tau(y(s)) = \alpha^{-1} x(s)$, $\tau(k_0) = k_0$. Moreover by the proof of Lemma 7.3, the highest weight space of $\text{Hom}(V, \mathbb{C})$ does not vanish on the highest weight space of V , i.e., $f(v) \neq 0$ for highest weight vectors f, v of $\text{Hom}(V, \mathbb{C})$, V , respectively. Now let W be a nonzero \mathcal{T} -submodule of the \mathcal{T} -module V via φ_s . Then W contains the highest weight space of V as shown in the last paragraph. This implies that W^\perp is a \mathcal{T} -submodule of $\text{Hom}(V, \mathbb{C})$ via φ_s and does not contain the highest weight space of $\text{Hom}(V, \mathbb{C})$. Recall $\text{Hom}(V, \mathbb{C})$ and V are isomorphic as \mathcal{T} -modules via φ_s . Thus $W^\perp = 0$. Therefore $W=V$ and the proof of the ‘if’ part of Theorem 1.15 (i) is completed.

7.2 Proof of the ‘if’ part of Theorem 1.18 (i), Theorem 1.21 (i)

We start with observations about the quantum algebra $U_q(sl_2)$. The quantum algebra $\mathcal{U} = U_q(sl_s)$ is the associative \mathbb{C} -algebra with 1 generated by X^\pm, K^\pm subject to the relations

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ KX^\pm K^{-1} &= q^{\pm 2} X^\pm, \\ [X^+, X^-] &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

$V(\ell)$ denotes the $(\ell + 1)$ -dimensional irreducible \mathcal{U} -module: $V(\ell) = \langle v_0, v_1, \dots, v_\ell \rangle$ and

$$\begin{aligned} K v_i &= q^{2i-\ell} v_i, \\ X^+ v_i &= [i + 1] v_{i+1}, \\ X^- v_i &= [\ell - i + 1] v_{i-1}, \end{aligned}$$

where $v_{-1} = v_{\ell+1} = 0$. We consider a finite-dimensional \mathcal{U} -module V that has the following weight-space decomposition:

$$V = \bigoplus_{i=0}^d U_i, \quad K|_{U_i} = q^{2i-d} \quad (0 \leq i \leq d). \quad (37)$$

Since V is completely reducible, we have

$$V = \bigoplus_{j=0}^{[d/2]} V^{(d-2j)},$$

where $V^{(\ell)}$ denotes the homogeneous component that is a direct sum of irreducible \mathcal{U} -modules isomorphic to $V(\ell)$; $V^{(\ell)}$ is allowed to be zero. Set

$$U_i^{(d-2j)} = U_i \cap V^{(d-2j)} \quad (0 \leq i \leq d, 0 \leq j \leq [d/2]).$$

Then

$$\begin{aligned} V^{(d-2j)} &= \bigoplus_{i=j}^{d-j} U_i^{(d-2j)} \quad (0 \leq j \leq [d/2]), \\ U_i &= \bigoplus_{j=0}^{i'} U_i^{(d-2j)} \quad (0 \leq i \leq d), \end{aligned}$$

where $i' = \min \{i, d - i\}$. For $j \leq i < d - j$, the mappings

$$\begin{aligned} X^+ : U_i^{(d-2j)} &\longrightarrow U_{i+1}^{(d-2j)}, \\ X^- : U_{i+1}^{(d-2j)} &\longrightarrow U_i^{(d-2j)} \end{aligned}$$

are inverses each other up to a nonzero scalar multiple and X^+ , X^- vanish on $U_{d-j}^{(d-2j)}, U_j^{(d-2j)}$, respectively. In particular,

$$U_j^{(d-2j)} = \ker (X^+)^{d-2j+1}|_{U_j} \quad (0 \leq j \leq [d/2]). \quad (38)$$

Lemma 7.5 *Let V be a finite-dimensional \mathcal{U} -module that satisfies (37). Let W be a subspace of V as a vector space. Assume W is invariant by the actions of X^+ and K :*

$$K W \subseteq W, \quad X^+ W \subseteq W.$$

If it holds that

$$\dim (W \cap U_i) = \dim (W \cap U_{d-i}) \quad (0 \leq i \leq d),$$

then $X^- W \subseteq W$, i.e., W is a \mathcal{U} -submodule.

Proof. Set $W_i = W \cap U_i$ ($0 \leq i \leq d$). Then since W is K -invariant, we have

$$W = \bigoplus_{i=0}^d W_i,$$

allowing W_i to be zero. Set $W_i^{(d-2j)} = W_i \cap U_i^{(d-2j)}$ ($0 \leq i \leq d$, $0 \leq j \leq [d/2]$). We claim

$$W_i = \bigoplus_{j=0}^i W_i^{(d-2j)} \quad (0 \leq i \leq [d/2]). \quad (39)$$

The claim holds for $i = 0$, since $W_0 \subseteq U_0 = U_0^{(d)}$. Suppose the claim holds up to i . Observe the mapping

$$(X^+)^{d-2i} : U_i \longrightarrow U_{d-i} \quad (0 \leq i \leq [d/2])$$

is a bijection. By $X^+W \subseteq W$, the image of W_i by $(X^+)^{d-2i}$ is included in W_{d-i} . Since $\dim W_i = \dim W_{d-i}$, the mapping

$$(X^+)^{d-2i} : W_i \longrightarrow W_{d-i} \quad (0 \leq i \leq [d/2])$$

is a bijection. Consider the mapping

$$(X^+)^{d-2i-1} : W_{i+1} \longrightarrow W_{d-i}.$$

The subspace X^+W_i of W_{i+1} is bijectively mapped onto W_{d-i} by $(X^+)^{d-2i-1}$. So we have

$$W_{i+1} = X^+W_i \oplus \ker (X^+)^{d-2i-1}|_{W_{i+1}}. \quad (40)$$

Since $W_i = \bigoplus_{j=0}^i W_i^{(d-2j)}$ by the induction hypothesis for the claim (39) and $X^+W_i^{(d-2j)} \subseteq W_{i+1}^{(d-2j)}$, we have

$$X^+W_i \subseteq \bigoplus_{j=0}^i W_{i+1}^{(d-2j)}.$$

On the other hand, since $\ker (X^+)^{d-2i-1}|_{U_{i+1}} = U_{i+1}^{(d-2i-2)}$ ($i+1 \leq [d/2]$) by (38), we have

$$\ker (X^+)^{d-2i-1}|_{W_{i+1}} = W_{i+1}^{(d-2i-2)}.$$

Thus by (40), we obtain $W_{i+1} \subseteq \bigoplus_{j=0}^{i+1} W_{i+1}^{(d-2j)}$. Since the opposite inclusion is obvious, the claim holds for $i+1$ and we finish the proof of the claim (39).

Since W_i is bijectively mapped onto W_{d-i} by $(X^+)^{d-2i}$ ($0 \leq i \leq [d/2]$), it follows from (39) that

$$W_{d-i} = \bigoplus_{j=0}^i W_{d-i}^{(d-2j)} \quad (0 \leq i \leq [d/2]), \quad (41)$$

$$W_{d-i}^{(d-2j)} = (X^+)^{d-2i} W_i^{(d-2j)} \quad (0 \leq j \leq i \leq [d/2]). \quad (42)$$

Define the subspace $W^{(d-2j)}$ by

$$W^{(d-2j)} = \bigoplus_{i=j}^{d-j} W_i^{(d-2j)} \quad (0 \leq j \leq [d/2]).$$

Then by (39), (41), we obtain

$$W = \bigoplus_{j=0}^{[d/2]} W^{(d-2j)}. \quad (43)$$

For $j \leq i < d-j$, the mappings $X^+ : U_i^{(d-2j)} \longrightarrow U_{i+1}^{(d-2j)}$ and $X^- : U_{i+1}^{(d-2j)} \longrightarrow U_i^{(d-2j)}$ are inverses each other up to a nonzero scalar multiple. The image of $W_i^{(d-2j)}$ by X^+ is contained in $W_{i+1}^{(d-2j)}$, in particular $\dim W_i^{(d-2j)} \leq \dim W_{i+1}^{(d-2j)}$ ($j \leq i < d-j$). On the other hand by (42), $\dim W_i^{(d-2j)} = \dim W_{d-i}^{(d-2j)}$ ($0 \leq j \leq i \leq [d/2]$). So $\dim W_i^{(d-2j)} = \dim W_{i+1}^{(d-2j)}$ ($j \leq i < d-j$). Therefore the mapping

$$X^+ : W_i^{(d-2j)} \longrightarrow W_{i+1}^{(d-2j)}$$

is a bijection for $j \leq i < d-j$ and the inverse of this mapping coincides with $X^-|_{W_{i+1}^{(d-2j)}}$ up to a nonzero scalar multiple. Thus we obtain $X^-W_{i+1}^{(d-2j)} = W_i^{(d-2j)}$ ($j < i+1 \leq d-j$). Since $X^-W_j^{(d-2j)} \subseteq X^-U_j^{(d-2j)} = 0$, it holds that $X^-W_i^{(d-2j)} \subseteq W$ ($0 \leq j \leq [d/2]$, $j \leq i \leq d-j$). Hence $X^-W \subseteq W$ by (43) and the proof of Lemma 7.5 is completed. \square

Proof of Theorem 1.21 (i). Theorem 1.21 (i) is well-known but we give a brief proof as a warm-up. We are in the case of $(\varepsilon, \varepsilon^*) = (0, 0)$ and given a \mathcal{T} -module

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

via φ_s . To prove Theorem 1.21 (i), it is enough to show that every \mathcal{T} -submodule W of V via φ_s is \mathcal{L} -invariant. The generators x, y, k, k^{-1} of \mathcal{T} act on V via φ_s as $\alpha s e_0^+, s^{-1} e_1^+, s k_0, s^{-1} k_1$, i.e., \mathcal{T} is embedded in \mathcal{L} via φ_s as the Borel subalgebra generated by $e_0^+, e_1^+, k_0^{\pm 1}$. V has weight-space decomposition as in (37):

$$V = \bigoplus_{i=0}^d U_i \quad (k_0|_{U_i} = q^{2i-d}),$$

where $d = \ell_1 + \cdots + \ell_n$. For a \mathcal{T} -submodule W of the \mathcal{T} -module V via φ_s , set $W_i = W \cap U_i$. Then

$$W = \bigoplus_{i=0}^d W_i.$$

Since the mapping $(e_0^+)^{d-2i} : U_i \longrightarrow U_{d-i}$ is a bijection and W_i is mapped into W_{d-i} by $(e_0^+)^{d-2i}$, we have $\dim W_i \leq \dim W_{d-i}$ ($0 \leq i \leq [d/2]$). Since the mapping $(e_1^+)^{d-2i} : U_{d-i} \longrightarrow$

U_i is a bijection and W_{d-i} is mapped into W_i by $(e_1^+)^{d-2i}$, we have $\dim W_{d-i} \leq \dim W_i$ ($0 \leq i \leq [d/2]$). Thus it holds that

$$\dim W_i = \dim W_{d-i} \quad (0 \leq i \leq d).$$

Consider the algebra homomorphism from the quantum algebra $\mathcal{U} = U_q(sl_2)$ to the $U_q(sl_2)$ -loop algebra \mathcal{L} that sends $X^+, X^-, K^{\pm 1}$ to $e_0^+, e_0^-, k_0^{\pm 1}$, respectively. Regard V as a \mathcal{U} -module via this algebra homomorphism. Then $X^+W \subseteq W, KW \subseteq W$. Since $\dim W_i = \dim W_{d-i}$ ($0 \leq i \leq d$), we have by Lemma 7.5 $X^-W \subseteq W$, i.e., $e_0^-W \subseteq W$. Similarly, consider the algebra homomorphism from \mathcal{U} to \mathcal{L} that sends $X^+, X^-, K^{\pm 1}$ to $e_1^+, e_1^-, k_1^{\pm 1}$, respectively. Regard V as a \mathcal{U} -module via this algebra homomorphism. Then the weight-space decomposition of this \mathcal{U} -module V is $V = \bigoplus_{i=0}^d U_{d-i}$ ($K|_{U_{d-i}} = q^{2i-d}$), where $V = \bigoplus_{i=0}^d U_i$ is the weight-space decomposition of the \mathcal{L} -module V . Since $\dim W_{d-i} = \dim W_i$ ($0 \leq i \leq d$) and $X^+W \subseteq W, KW \subseteq W$, we have by Lemma 7.5, $X^-W \subseteq W$, i.e., $e_1^-W \subseteq W$. Thus W is \mathcal{L} -invariant and the proof of Theorem 1.21 (i) is completed. \square

We are now ready to prove the ‘if’ part of Theorem 1.18 (i).

Proof of the ‘if’ part of Theorem 1.18 (i). We are in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$ and given a \mathcal{T} -module

$$V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

via φ_s such that $-s^{-2} \notin S(\ell_i, a_i)$ ($1 \leq i \leq n$) and the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ is in general position. We want to show that the \mathcal{T} -module V is irreducible. Note that $P_{V(\ell)}(s^{-2}) \neq 0$, $P_{V(\ell_i, a_i)}(s^{-2}) \neq 0$ by Theorem 5.2, so

$$\sigma_d(V) \neq 0 \tag{44}$$

by Remark 1.10, where $d = \ell + \ell_1 + \cdots + \ell_n$. We may assume $n \geq 1$, otherwise $V = V(\ell)$ is obviously irreducible as a \mathcal{T} -module, since $\sigma_\ell(V) \neq 0$. Consider i_0 such that $a_{i_0}q^{\ell_{i_0}-1}$ is maximal in the set of scalars $a_iq^{\ell_i-1}$ ($1 \leq i \leq n$) with respect to the partial ordering introduced in (34) in Section 7.1. Among such i_0 s, choose one that has the smallest ℓ_{i_0} . Since $\{S(\ell_i, a_i)\}_{i=1}^n$ is in general position, the ordering of $V(\ell_i, a_i)$ ($1 \leq i \leq n$) in the tensor product of V does not matter about the isomorphism class of V as an \mathcal{L}' -module and consequently as a \mathcal{T} -module via φ_s . So we may assume $i_0 = n$. Then

$$a_nq^{\ell_n+1} \notin S(\ell_i, a_i) \quad (1 \leq i \leq n), \tag{45}$$

$$\ell_n \leq \ell_i \text{ if } a_nq^{\ell_n-1} \in S(\ell_i, a_i). \tag{46}$$

We proceed by induction on $\dim V$ to prove that V is irreducible as a \mathcal{T} -module. Set

$$V' = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_{n-1}, a_{n-1}).$$

Then $V = V' \otimes V(\ell_n, a_n)$. Since the \mathcal{L} -module $V(\ell_n, a_n)$ is by Lemma 5.6 embedded in the \mathcal{L} -module $V(\ell_n - 1, a_nq^{-1}) \otimes V(1, a_nq^{\ell_n-1})$ as the \mathcal{L} -submodule spanned by the highest weight space, the \mathcal{L}' -module V can be regarded as embedded in the \mathcal{L}' -module

$$\tilde{V} = (V' \otimes V(\ell_n - 1, a_nq^{-1})) \otimes V(1, a_nq^{\ell_n-1}),$$

sharing the highest weight space in common. We understand $V' \otimes V(\ell_n - 1, a_n q^{-1}) = V'$ if $\ell_n = 1$. To apply Proposition 6.1 to \tilde{V} , we check the prerequisites for it, namely that $V' \otimes V(\ell_n - 1, a_n q^{-1})$ is irreducible as a \mathcal{T} -module via φ_s , and that $\sigma_d(\tilde{V}) \neq 0$ holds, where $d = \ell + \ell_1 + \cdots + \ell_n$ is the diameter of the \mathcal{T} -module \tilde{V} via φ_s . Observe that $S(\ell_n - 1, a_n q^{-1}) = S(\ell_n, a_n) \setminus \{a_n q^{\ell_n - 1}\}$. So $-s^{-2} \notin S(\ell_i, a_i)$ ($1 \leq i \leq n - 1$) and $-s^{-2} \notin S(\ell_n - 1, a_n q^{-1})$. Moreover the multi-set $\{S(\ell_n, a_n)\}_{i=1}^{n-1} \cup \{S(\ell_n - 1, a_n q^{-1})\}$ of q -strings associated with $V' \otimes V(\ell_n - 1, a_n q^{-1})$ is in general position by (45), (46). Therefore by induction on dimension, $V' \otimes V(\ell_n - 1, a_n q^{-1})$ is irreducible as a \mathcal{T} -module via φ_s . Since $P_V(s^{-2}) \neq 0$ as we observed before, and since $P_{\tilde{V}}(\lambda) = P_V(\lambda)$ by Theorem 5.2, we have $P_{\tilde{V}}(s^{-2}) \neq 0$, i.e., $\sigma_d(\tilde{V}) \neq 0$. Thus the prerequisites are satisfied for Proposition 6.1 to be applied to \tilde{V} . On the other hand, the conclusion of Proposition 6.1

$$P_{V' \otimes V(\ell_n - 1, a_n q^{-1})}(-a_n q^{\ell_n + 1}) = 0$$

fails by Theorem 5.2, since $a_n q^{\ell_n + 1} \notin S(\ell_i, a_i)$ ($1 \leq i \leq n$) by (45). This implies that any nonzero \mathcal{T} -submodules W of \tilde{V} via φ_s contains the highest weight space of \tilde{V} . Since the \mathcal{T} -module V via φ_s is embedded in the \mathcal{T} -module \tilde{V} via φ_s , sharing the highest weight space in common, we conclude that any nonzero \mathcal{T} -submodule W of V contains the highest weight space of V .

Let W be a minimal \mathcal{T} -submodule of the \mathcal{T} -module V via φ_s . Note that W is irreducible as a \mathcal{T} -module. Let $V = \bigoplus_{i=0}^d U_i$ denote the weight-space decomposition of the \mathcal{T} -module V via φ_s . Then

$$W = \bigoplus_{i=0}^d W_i, \quad W_i = W \cap U_i \quad (1 \leq i \leq n).$$

By what we just proved in the last paragraph, we have $W_0 \neq 0$. Moreover $W_d \neq 0$ by (44). Since $\dim U_0 = \dim U_d = 1$, we obtain $W_0 = U_0$, $W_d = U_d$. We claim

$$\dim W_i = \dim W_{d-i} \quad (0 \leq i \leq d). \quad (47)$$

Let \mathcal{A} denote the TD-algebra for $(\varepsilon, \varepsilon^*) = (1, 0)$. Consider $\varphi_s \circ \iota_t : \mathcal{A} \longrightarrow \mathcal{L}'$ and regard V as an \mathcal{A} -module via $\varphi_s \circ \iota_t$. By Theorem 1.11 and (13), the generators z, z^* , of \mathcal{A} act on W as a TD-pair, if we choose t suitably. The split decomposition of W for the TD-pair coincides with the weight-space decomposition of W . Thus we obtain (47) by [3, Corollary 5.7].

The generators x, y, k, k^{-1} of \mathcal{T} act on V via φ_s as $\alpha(se_0^+ + s^{-1}e_1^-k_1)$, $s^{-1}e_1^+$, sk_0 , $s^{-1}k_1$ respectively. Consider the algebra homomorphism from \mathcal{U} to \mathcal{L}' that sends $X^+, X^-, K^{\pm 1}$ to $e_1^+, e_1^-, k_1^{\pm 1}$. Regard V as a \mathcal{U} -module via this algebra homomorphism. Then the weight-space decomposition of this \mathcal{U} -module V is $V = \bigoplus_{i=0}^d U_{d-i}$ ($K|_{U_{d-i}} = q^{2i-d}$), where $V = \bigoplus_{i=0}^d U_i$ is the weight space-decomposition of the \mathcal{T} -module V via φ_s . Since $\dim W_{d-i} = \dim W_i$ ($0 \leq i \leq d$) by (47) and $X^+W \subseteq W$, $KW \subseteq W$, we have by Lemma 7.5 $X^-W \subseteq W$, i.e., $e_1^-W \subseteq W$. Since $xW \subseteq W$, i.e., $(se_0^+ + s^{-1}e_1^-k_1)W \subseteq W$, we obtain $e_0^+W \subseteq W$ by $e_1^-k_1W \subseteq W$. Thus W is \mathcal{L}' -invariant. Recall that we have already shown that W contains the highest weight space U_0 of the \mathcal{T} -module V via φ_s . By the following lemma, we obtain $W = V$ and the 'if' part of Theorem 1.18 (i) is completed.

Lemma 7.6 Assume that a multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings is in general position. Consider the \mathcal{L}' -module

$$V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

and let

$$V = \bigoplus_{i=0}^d U_i, \quad k_0|_{U_i} = q^{2i-d}$$

be the eigenspace decomposition of k_0 , where $d = \ell + \ell_1 + \cdots + \ell_n$. If W is an \mathcal{L}' -submodule of V and contains U_0 , then $W = V$.

Proof. Set

$$V' = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

and let \mathcal{B} denote the subalgebra of \mathcal{L}' generated by e_0^+ , e_1^+ , $k_0^{\pm 1}$. Note that V' is irreducible as an \mathcal{L}' -module, since it is already irreducible as a \mathcal{B} -module by Theorem 1.21 (i). We may assume $\ell \geq 1$, since if $\ell = 0$, then $V = V'$ and the \mathcal{L}' -module V is irreducible.

Let $V(\ell) = \langle v_0, v_1, \dots, v_\ell \rangle$ be a standard basis as an \mathcal{L}' -module: $e_0^+ v_i = 0$, $e_1^+ v_i = [\ell - i + 1] v_{i-1}$, $e_1^- v_i = [i + 1] v_{i+1}$, $k_0 v_i = q^{2i-\ell} v_i$ ($0 \leq i \leq \ell$), where $v_{-1} = v_{\ell+1} = 0$. Then

$$V = \bigoplus_{i=0}^{\ell} \langle v_i \rangle \otimes V'.$$

We show $W \supseteq \langle v_i \rangle \otimes V'$ ($0 \leq i \leq \ell$) by induction on i . For $i = 0$, some element

$$v_0 \otimes v' \quad (V' \ni v' \neq 0)$$

is contained in W by $W \supseteq U_0$. Since $e_0^+(v_0 \otimes v') = q^{-\ell} v_0 \otimes (e_0^+ v')$, $e_1^+(v_0 \otimes v') = q^\ell v_0 \otimes (e_1^+ v')$ and $k_0^{\pm 1}(v_0 \otimes v') = q^{\pm \ell} v_0 \otimes (k_0^{\pm 1} v')$, it follows from $\mathcal{B}W \subseteq W$ that $v_0 \otimes (e_0^+ v')$, $v_0 \otimes (e_1^+ v')$, $v_0 \otimes (k_0^{\pm 1} v')$ are contained in W . Since the elements e_0^+ , e_1^+ , $k_0^{\pm 1}$ generate \mathcal{B} , we obtain

$$\langle v_0 \rangle \otimes \mathcal{B}v' \subseteq W.$$

Since V' is irreducible as a \mathcal{B} -module by Theorem 1.21 (i), we have $\mathcal{B}v' = V'$ so $v_0 \otimes V' \subseteq W$. Suppose that $\langle v_i \rangle \otimes V' \subseteq W$. Choose a nonzero element v' from V' . Then $e_1^-(v_i \otimes v') = [i + 1] v_{i+1} \otimes (k_1^{-1} v') + v_i \otimes (e_1^- v')$. Since $e_1^-(v_i \otimes v')$ and $v_i \otimes (e_1^- v')$ are contained in W , we have

$$v_{i+1} \otimes v'' \in W,$$

where $v'' = k_1^{-1} v' \neq 0$. So $e_0^+(v_{i+1} \otimes v'')$, $e_1^+(v_{i+1} \otimes v'')$, $k_0(v_{i+1} \otimes v'')$ are contained in W . Since $e_0^+(v_{i+1} \otimes v'') = q^{2i+2-\ell} v_{i+1} \otimes (e_0^+ v'')$, $e_1^+(v_{i+1} \otimes v'') = [\ell - i] v_i \otimes v'' + q^{\ell-2i-2} v_{i+1} \otimes (e_1^+ v'')$, $k_0(v_{i+1} \otimes v'') = q^{2i+2-\ell} v_{i+1} \otimes (k_0 v'')$, it follows from $v_i \otimes v'' \in W$ that

$$v_{i+1} \otimes (e_0^+ v''), v_{i+1} \otimes (e_1^+ v''), v_{i+1} \otimes (k_0 v'')$$

are all contained in W . So

$$\langle v_{i+1} \rangle \otimes \mathcal{B}v'' \subseteq W.$$

Since $\mathcal{B}v'' = V'$, we have $\langle v_{i+1} \rangle \otimes V' \subseteq W$. This completes the proof of Lemma 7.6. \square

7.3 Proof of part (iii)

The part (iii) of Theorem 1.15, Theorem 1.18, Theorem 1.21 follows from the part (i) together with Theorem 1.9', Theorem 5.2, and some combinatorial observations as in Lemma 1.14; we prove Lemma 1.14 at the end of this subsection separately. Let s and d be a nonzero scalar and a positive integer respectively, chosen arbitrarily. We are given a polynomial $P(\lambda)$ in \mathcal{P}_d^s , i.e., $P(\lambda)$ is a monic polynomial of degree d such that $P(\varepsilon s^{-2} + \varepsilon^* s^2) \neq 0$. We want to construct an irreducible \mathcal{T} -module V via φ_s such that the Drinfel'd polynomial $P_V(\lambda)$ coincides with $P(\lambda)$. Let $\lambda_1, \lambda_2, \dots, \lambda_d$ denote the roots of $P(\lambda)$, allowing repetition.

If $(\varepsilon, \varepsilon^*) = (1, 1)$, let Ω_i denote the set of solutions of

$$\lambda_i + \zeta + \zeta^{-1} = 0$$

for ζ . We understand that Ω_i is a multi-set if $\lambda_i = \pm 2$. So $|\Omega_i| = 2$ ($1 \leq i \leq d$). Set

$$\Omega = \bigcup_{i=1}^d \Omega_i$$

as a multi-set. Then $|\Omega| = 2d$ as a multi-set. By Lemma 1.14, there exists a multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings strongly in general position such that

$$\Omega = \bigcup_{i=1}^n (S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1}))$$

as multi-sets. Since $|S(\ell_i, a_i)| = \ell_i$, we have $d = \ell_1 + \dots + \ell_n$. The \mathcal{T} -module

$$V = V(\ell_1, a_1) \otimes \dots \otimes V(\ell_n, a_n)$$

via φ_s has Drinfel'd polynomial

$$P_V(\lambda) = \prod_{i=1}^n P_{V(\ell_i, a_i)}(\lambda)$$

by Theorem 5.2, where

$$P_{V(\ell_i, a_i)}(\lambda) = \prod_{\zeta \in S(\ell_i, a_i)} (\lambda + \zeta + \zeta^{-1}).$$

Thus $P_V(\lambda) = P(\lambda)$. Since $P(s^{-2} + s^2) \neq 0$, we have $-s^2 \notin S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})$ ($1 \leq i \leq n$). So by Theorem 1.15 (i), the \mathcal{T} -module V via φ_s is irreducible.

If $(\varepsilon, \varepsilon^*) = (1, 0)$, set

$$\Omega = \{-\lambda_i \mid \lambda_i \neq 0, 1 \leq i \leq d\}$$

as a multi-set. We may assume that $\Omega = \{-\lambda_i \mid \ell + 1 \leq i \leq d\}$ and $\lambda_1 = \dots = \lambda_\ell = 0$, allowing $\ell = 0$. It is well-known and easy to show that there exists a multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q -strings in general position such that

$$\Omega = \bigcup_{i=1}^n S(\ell_i, a_i)$$

as multi-sets. Since $|S(\ell_i, a_i)| = \ell_i$, we have $d - \ell = \ell_1 + \cdots + \ell_n$. The \mathcal{T} -module

$$V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

via φ_s has Drinfel'd polynomial

$$P_V(\lambda) = \lambda^\ell \prod_{i=1}^n P_{V(\ell_i, a_i)}(\lambda)$$

by Theorem 5.2, where

$$P_{V(\ell_i, a_i)}(\lambda) = \prod_{c \in S(\ell_i, a_i)} (\lambda + c).$$

Thus $P_V(\lambda) = P(\lambda)$. Since $P(s^{-2}) \neq 0$, we have $-s^{-2} \notin S(\ell_i, a_i)$ ($1 \leq i \leq n$). So by Theorem 1.18 (i), the \mathcal{T} -module V via φ_s is irreducible.

If $(\varepsilon, \varepsilon^*) = (0, 0)$, set

$$\Omega = \{-\lambda_1, \lambda_2, \dots, -\lambda_d\},$$

as a multi-set. Since $P(\varepsilon s^{-2} + \varepsilon^* s^2) = P(0) \neq 0$, we have $\lambda_i \neq 0$ ($1 \leq i \leq n$). There exists a multi-set of q -strings $\{S(\ell_i, a_i)\}_{i=1}^n$ in general position such that

$$\Omega = \bigcup_{i=1}^n S(\ell_i, a_i)$$

as multi-set. Since $|S(\ell_i, a_i)| = \ell_i$, we have $d = \ell_1 + \cdots + \ell_n$. The \mathcal{T} -module

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

via φ_s has Drinfel'd polynomial

$$P_V(\lambda) = \prod_{i=1}^n P_{V(\ell_i, a_i)}(\lambda)$$

by Theorem 5.2, where

$$P_{V(\ell_i, a_i)}(\lambda) = \prod_{c \in S(\ell_i, a_i)} (\lambda + c).$$

Thus $P_V(\lambda) = P(\lambda)$. By Theorem 1.21 (i), the \mathcal{T} -module V via φ_s is irreducible. This completes the proof of the part (iii) for Theorem 1.15, Theorem 1.18, Theorem 1.21.

Proof of Lemma 1.14. We proceed by induction on $|\Omega|$, where $|\Omega|$ denotes the number of elements in Ω , counting the multiplicities. Recall the partial ordering (34) on $\mathbb{C} \setminus \{0\}$ introduced in Section 7.1:

$$a \leq b \Leftrightarrow b = a q^{2i} \text{ for some integer } i \geq 0.$$

Choose a maximal element c in Ω with respect to this partial ordering. Note that c^{-1} is minimal in Ω . Set

$$\Omega' = \Omega \setminus \{c, c^{-1}\}$$

as multi-sets of nonzero scalars. Then by induction, there exists a multi-set $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ of q -strings strongly in general position such that

$$\Omega' = \bigcup_{i=1}^{n'} (S(\ell'_i, a'_i) \cup S(\ell'_i, a'^{-1}_i))$$

as multi-sets of nonzero scalars. Moreover such a multi-set $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ of q -strings is uniquely determined by Ω' up to equivalence. Observe that the union $S(\ell'_i, a'_i) \cup \{c\}$ (resp. $S(\ell'_i, a'_i) \cup \{c^{-1}\}$) as a multi-set of nonzero scalars is a q -string if and only if $c = a'_i q^{\ell'_i+1}$ (resp. $c^{-1} = a'_i q^{-\ell'_i-1}$), in which case

$$S(\ell'_i, a'_i) \cup \{c\} = S(\ell'_i + 1, a'_i q) \quad (\text{resp. } S(\ell'_i, a'_i) \cup \{c^{-1}\} = S(\ell'_i + 1, a'_i q^{-1})).$$

If there exist i 's such that either $c = a'_i q^{\ell'_i+1}$ or $c^{-1} = a'_i q^{-\ell'_i-1}$, choose one among such i 's that has the largest ℓ'_i . By rearranging the ordering of the q -strings, we may assume $i = n'$. By replacing $a'_{n'}$ by $a'^{-1}_{n'}$ if $c^{-1} = a'_{n'} q^{-\ell'_{n'}-1}$, we may assume $c = a'_{n'} q^{\ell'_{n'}+1}$. Thus $c = a'_{n'} q^{\ell'_{n'}+1}$ is maximal in Ω and if $c = a'_i q^{\ell'_i+1}$ or $c^{-1} = a'_i q^{-\ell'_i-1}$ holds for some i , then $\ell'_n \geq \ell'_i$. In this case, define q -strings $S(\ell_i, a_i)$ ($1 \leq i \leq n'$) by

$$S(\ell_i, a_i) = S(\ell'_i, a'_i) \quad (1 \leq i \leq n' - 1), \quad (48)$$

$$S(\ell_{n'}, a_{n'}) = S(\ell'_{n'} + 1, a'_{n'} q). \quad (49)$$

Then the multi-set $\{S(\ell_i, a_i)\}_{i=1}^{n'}$ of q -strings is strongly in general position and

$$\Omega = \bigcup_{i=1}^{n'} (S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1}))$$

as multi-set of nonzero scalars.

If there exist no i 's such that either $c = a'_i q^{\ell'_i+1}$ or $c^{-1} = a'_i q^{-\ell'_i-1}$, then define q -strings $S(\ell_i, a_i)$ ($1 \leq i \leq n' + 1$) by

$$S(\ell_i, a_i) = S(\ell'_i, a'_i) \quad (1 \leq i \leq n'), \quad (50)$$

$$S(\ell_{n'+1}, a_{n'+1}) = S(1, c). \quad (51)$$

Then the multi-set $\{S(\ell_i, a_i)\}_{i=1}^{n'+1}$ of q -strings is strongly in general position and

$$\Omega = \bigcup_{i=1}^{n'+1} (S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1}))$$

as multi-sets of nonzero scalars. In any case, there exists a desired multi-set of q -strings.

Next we show the uniqueness of such a multi-set of q -strings up to equivalence. Let $\{S(m_i, b_i)\}_{i=1}^n$ be a multi-set of q -strings strongly in general position such that

$$\Omega = \bigcup_{i=1}^n (S(m_i, b_i) \cup S(m_i, b_i^{-1})).$$

Then the maximal element c of Ω , which was chosen in the course of the construction of a desired multi-sets of q -strings, belongs to either $S(m_i, b_i)$ or $S(m_i, b_i^{-1})$ for some i . Among such i 's, choose one that has the smallest m_i . We may assume $i = n$ and $c \in S(m_n, b_n)$ by rearranging the ordering of the q -strings and replacing b_n by b_n^{-1} if necessary. Thus $b_n q^{m_n-1}$ is the maximal element c and $m_n \leq m_i$ holds if $c \in S(m_i, b_i)$ or $c \in S(m_i, b_i^{-1})$, i.e., $c = b_i q^{m_i-1}$ or $c^{-1} = b_i q^{-m_i+1}$.

If $m_n \geq 2$, then the multi-set

$$\{S(m_i, b_i)\}_{i=1}^{n-1} \cup \{S(m_n - 1, b_n q^{-1})\}$$

of q -strings is strongly in general position and covers the multi-set $\Omega' = \Omega \setminus \{c, c^{-1}\}$ of nonzero scalars as the union of $S(m_i, b_i)$, $S(m_i, b_i^{-1})$ ($1 \leq i \leq n-1$) and $S(m_n - 1, b_n q^{-1})$, $S(m_n - 1, b_n^{-1} q)$. Such a multi-set of q -strings is unique up to equivalence by induction. So the multi-set $\{S(m_i, b_i)\}_{i=1}^{n-1} \cup \{S(m_n - 1, b_n q^{-1})\}$ of q -strings is equivalent to $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$, the one which was chosen in the course of the construction of a desired multi-sets of q -strings. Observe that $c = (b_n q^{-1}) q^{(m_n-1)+1}$ and $m_n - 1 \geq m_i$ if $c = b_i q^{m_i+1}$ or $c^{-1} = b_i q^{-m_i-1}$ for some i ($1 \leq i \leq n-1$), since $S(m_n, b_n)$ includes either $S(m_i, b_i)$ or $S(m_i, b_i^{-1})$ for such an i . Thus we have $n = n'$ and we may assume

$$\begin{aligned} S(m_i, b_i) &= S(\ell'_i, a'_i) \quad (1 \leq i \leq n-1), \\ S(m_n - 1, b_n q^{-1}) &= S(\ell'_n, a'_n). \end{aligned}$$

By (48), (49), the multi-set $\{S(m_i, b_i)\}_{i=1}^n$ of q -strings is equivalent to the one we constructed by means of $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$.

If $m_n = 1$, then $S(m_n, b_n) = \{c\}$ and the multi-set

$$\{S(m_i, b_i)\}_{i=1}^{n-1}$$

of q -strings is strongly in general position and covers the multi-set $\Omega' = \Omega \setminus \{c, c^{-1}\}$ as a union of $S(m_i, b_i)$, $S(m_i, b_i^{-1})$ ($1 \leq i \leq n-1$). Such a multi-set of q -strings is unique up to equivalence. Observe that there exist no i 's ($1 \leq i \leq n-1$) such that either $c = b_i q^{m_i+1}$ or $c^{-1} = b_i q^{-m_i-1}$, since otherwise $S(m_n, b_n) \cup S(m_i, b_i)$ or $S(m_n, b_n^{-1}) \cup S(m_i, b_i)$ would be a q -string for such an i . Thus we have $n' = n-1$ and we may assume

$$S(m_i, b_i) = S(\ell'_i, a'_i) \quad (1 \leq i \leq n-1).$$

By (50), (51), the multi-set $\{S(m_i, b_i)\}_{i=1}^n$ of q -strings is equivalent to the one we constructed by means of $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$. This completes the proof of Lemma 1.14. \square

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